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On the compatibility relation for the Föppl–von Kármán plate equations

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ABSTRACT

By using a coordinate-free approach we propose a new derivation of the compatibility equation for the Föppl–von Kármán nonlinear plate theory.

1. Introduction

Historically, the Föppl–von Kármán (FvK) plate theory [1,2] has invariably proved to be a versatile model able to capture a plethora of different physical phenomena (see [3] and the references therein). The derivations of these equations given in many standard texts (e.g., [4–7]) are confined to the use of Cartesian coordinates that tend to be unnecessarily restrictive and fail to establish a deeper connection with the traditional theory of linear elasticity. It is this latter aspect that we want to revisit here and show how the classical compatibility relation for the linearised strain tensor can be transposed to the nonlinear case of the FvK plate kinematics. This will be done without any reference to a specific system of coordinates.

In Section 1 we present a succinct overview of those elements in the FvK theory that will be needed for our subsequent purposes; a complete tensorial derivation of the plate model in the spirit of this note can be found in [8,9]. The elastic plate of thickness $h > 0$ considered in what follows is assumed to occupy the region $\Omega \times [-h/2, h/2] \subset \mathbb{R}^2 \times \mathbb{R}$ and is subjected to a transverse pressure p (and possibly other in-plane loading conditions that will not affect our discussion). The position of points inside this domain is identified by an arbitrary system of coordinates defined by the set of linearly independent vectors $\mathcal{B} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$. For convenience, it will be assumed that the origin is situated in the plate midplane (Ω) and $\boldsymbol{g}_3$ is a unit vector perpendicular to it. A consequence of this fact is that if $\{\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3\}$ represents the reciprocal basis of $\mathcal{B}$ then $\boldsymbol{g}_3 = \boldsymbol{g}^3$. The coordinates with respect to the chosen basis will be labelled $\theta^1, \theta^2, \theta^3$ and, in line with the standard notation in the literature (and the assumptions already made), we shall set $\theta^3 = z$. Thus,

$$\nabla = \boldsymbol{g}^i \frac{\partial}{\partial \theta^i} = \boldsymbol{g}^\alpha \frac{\partial}{\partial \theta^\alpha} + \boldsymbol{g}^3 \frac{\partial}{\partial z} \equiv \nabla^* + \boldsymbol{g}^3 \frac{\partial}{\partial z},$$

where ∇^* is the projection of the gradient operator onto the plate midplane. In using the indicial notation above, it is assumed that the Greek indices range over the values $\{1, 2\}$, while for Latin indices this is changed to $\{1, 2, 3\}$.

One of the key assumptions in the FvK plate theory is that the displacement field $\boldsymbol{u} \equiv \boldsymbol{u}(\theta^i)$ has a special form based on the Love–Kirchhoff hypothesis (normals to the originally flat plate midplane remain straight after the deformation and continue to be perpendicular to the deformed midsurface); in particular,

$$\boldsymbol{u}(\theta^i) = \boldsymbol{v}(\theta^\alpha) - z\nabla^* w + w(\theta^\alpha)\boldsymbol{g}_3, \tag{1}$$

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where $\boldsymbol{v}$ represents the in-plane displacement field ($\boldsymbol{v} = \boldsymbol{g}^\alpha v_\alpha$) and w characterises the transverse displacements experienced by the plate midplane. Another simplification introduced by the FvK formalism is related to a special approximation of the Lagrangian strain tensor $\boldsymbol{E}$. Recall that

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} - \boldsymbol{I}_3) = \frac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^{\mathrm{T}} + \boldsymbol{H}^{\mathrm{T}} \cdot \boldsymbol{H}), \tag{2}$$

where $\boldsymbol{H} := \boldsymbol{u} \otimes \nabla$ is the displacement gradient and $\boldsymbol{F} = \boldsymbol{I}_3 + \boldsymbol{H}$ represents the deformation gradient ($\boldsymbol{I}_3$ is the three-dimensional identity tensor). Using (1), it follows that the (plane stress) Lagrangian tensor in the three-dimensional domain occupied by the plate can be approximated as

$$\boldsymbol{E} \simeq \frac{1}{2}(\nabla^* \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \nabla^*) - z\nabla^* \otimes \nabla^* w + \frac{1}{2}(\nabla^* w) \otimes (\nabla^* w). \tag{3}$$

Henceforth all differential operations will be carried out on fields defined on $\Omega \subset \mathbb{R}^2$, so it seems desirable to drop the asterisk on the del operator as there is no risk of confusion.

Since the plate is assumed to be very thin, and owing to the fact that the explicit dependence on the transverse coordinate z is fairly simple, it is customary in plate theory to define stress resultants by through-thickness integration. For example, the so-called membrane stress tensor $\boldsymbol{N} = (N^{\alpha\beta})$ is given by

$$N^{\alpha\beta}(\theta^1, \theta^2) := \int_{-h/2}^{h/2} \Pi^{\alpha\beta}(\theta^1, \theta^2, z)\,\mathrm{d}z, \tag{4}$$

where $\Pi^{\alpha\beta}$ are the contravariant components of the symmetric second Piola–Kirchhoff stress tensor. By the same token, one can use (3) to introduce the average strain measures

$$\boldsymbol{m} := \frac{12}{h^3}\int_{-h/2}^{h/2} z\boldsymbol{E}\,\mathrm{d}z, \qquad \boldsymbol{n} := \frac{1}{h}\int_{-h/2}^{h/2} \boldsymbol{E}\,\mathrm{d}z, \tag{5}$$

describing bending and, respectively, elongational deformation experienced by the plate; note that with these notations, $\boldsymbol{E} = \boldsymbol{n} + z\boldsymbol{m}$.

If E is the Young's modulus of the plate and ν denotes its Poisson ratio, Hookes's law for plane stress assumes the form

$$\Pi^{\alpha\beta} = \frac{E}{1+\nu}\left[g^{\alpha\gamma}g^{\beta\delta}E_{\gamma\delta} + \frac{\nu}{1-\nu}g^{\alpha\beta}g^{\gamma\delta}E_{\gamma\delta}\right],$$

where $g^{\alpha\beta} := \boldsymbol{g}^\alpha \cdot \boldsymbol{g}^\beta$. When used in conjunction with (4) and (5), this yields

$$\boldsymbol{N} = C\left\{\left(\frac{1-\nu}{2}\right)\Big[\nabla \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \nabla + (\nabla w) \otimes (\nabla w)\Big] + \nu\left(\nabla \cdot \boldsymbol{v} + \frac{1}{2}|\nabla w|^2\right)\boldsymbol{I}\right\}, \tag{6}$$

where $C := Eh/(1-\nu^2)$ represents the *in-plane stiffness* of the plate and $\boldsymbol{I}$ is the two-dimensional identity tensor.

One of the standard forms of the FvK equations ([4], for instance) can be stated as

$$D\nabla^4 w - \nabla \cdot (\boldsymbol{N} \cdot \nabla w) = p, \tag{7a}$$
$$\nabla \cdot \boldsymbol{N} = \boldsymbol{0}, \tag{7b}$$

where $D := Eh^3/12(1-\nu^2)$ is the *bending stiffness* of the plate; this represents a system of three nonlinear partial differential equations in w and the components of the in-plane displacement field $\boldsymbol{v}$.

The aim of this short note is to provide a new derivation of a well-known equivalent form of these equations involving only the transverse displacement w together with a stress potential (ϕ, say) (e.g., see [3,5–7]). No direct reference is made to either rectangular or polar coordinates since we employ an invariant approach.

2. The compatibility relation for the FvK plate theory

The nonlinear membrane tensor $\boldsymbol{N}$ in (6) is represented by using a suitably defined stress potential ϕ so that Eq. (7b) is automatically satisfied. This still leaves us with identifying an appropriate equation for ϕ. As we shall see shortly, such an equation will follow from using an argument similar to the compatibility conditions for the usual strain tensor in linear elasticity.

The stress potential is introduced by postulating

$$\boldsymbol{N} = (\nabla^2\phi)\boldsymbol{I} - \nabla \otimes \nabla\phi, \tag{8}$$

where the function $\phi \equiv \phi(\theta^\alpha)$ will be one of the unknowns in the equations that are to be derived. It is a simple exercise to check that with this choice (7b) is identically satisfied. As for the first equation in (7), this can now be recast by noticing