Applied Mathematics Letters 25 (2012) 1320–1326

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Existence and blow up for a nonlinear hyperbolic equation with anisotropy[☆]

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ARTICLE INFO

Article history:
Received 24 June 2011
Received in revised form 28 November 2011
Accepted 28 November 2011

Keywords:
Anisotropy
Nonlinear hyperbolic equation
Blow up

ABSTRACT

In this paper, we study a nonlinear wave equation with anisotropy and a source term. Under some appropriate assumptions on the parameters, and with certain initial data, we obtain several results on the existence of local solutions and the blow up of solutions in finite time.

1. Introduction

We study the following wave equation with anisotropy

$$\begin{cases} u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) - \Delta u_t = g(u), & \text{in } [0,T)\times\Omega, \\ u = 0, & \text{on } [0,T)\times\partial\Omega, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & \text{in } \bar{\Omega}. \end{cases} \tag{1.1}$$

Here $p_i \geq 2$, $i = 1, \ldots, n$, $T > 0$, and Ω is a bounded open subset of $R^n (n \geq 1)$ with a smooth boundary $\partial\Omega$. The function $g(u) = u|u|^{\sigma-2}$ is a polynomial source.

Problems related to the well-posedness of solutions to quasilinear wave equations

$$u_{tt} - \sum_{i=1}^{n}\{\sigma_i(u_{x_i})\}_{x_i} - \Delta u_t = f(x,t) \quad \text{in } (0,T)\times\Omega$$

have attracted a great deal of attention in the literature. Some important results can be found in [1–8]. Nonlinear hyperbolic equations of the type (1.1) have been investigated in the papers [9–16], just to cite a few. In these papers, by using Faedo–Galerkin approximation together with a combination of the compactness and the monotonicity methods, the authors obtained the existence of a global solution and derived decay results for the global solution. Problem (1.1) with $p_i = 2$ has been studied by many authors, notable among them is [17–20]. Gao and Ma [10] investigated the existence and asymptotic behavior of the solutions to problem (1.1) with $p_i = p > 2$. Sango [13] studied the existence of the solutions of the following

☆ This work is supported by the National Science Foundation of China (Nos. 60974034, 61174082).

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0893-9659/$ – see front matter
doi:10.1016/j.aml.2011.11.037

initial boundary value problem

$$\begin{cases} u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) - \Delta u_t + g(x,u) = f(x,t), & \text{in } [0,T)\times\Omega, \\ u = 0, & \text{on } [0,T)\times\partial\Omega, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & \text{in } \bar{\Omega}, \end{cases} \tag{1.2}$$

where $p_i \geq 2$, $\Omega \subset R^n (n \geq 1)$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ the function $g(x,u)$ satisfies $|g(x,u)| \leq a|u|^{\sigma-1}$, $f(x,t)$ is a given function. The author proved the global existence of the solution to problem (1.2) when the parameters satisfy $1 < \sigma < \min\{p^*, p_1, \ldots, p_n\}$, and the function g satisfies $g(x,u)u \leq \delta G(x,u)$, where $G(x,u) = \int_0^u g(x,s)ds$, and $p^* = \frac{n\bar{p}}{n-\bar{p}}$, $\frac{1}{\bar{p}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}$. Closely related results may be found in the references cited in this paper and in [21,22].

It is well known that in (1.2), the damping term $-\Delta u_t$ plays a critical role in establishing the existence of solutions, and the attractive source term $g(u)$, i.e. $g(u)u \geq 0$, stabilizes the solution, while the forcing source $g(u)$, i.e. $g(u)u < 0$, destabilizes the solution and causes a finite time blow up.

Inspired by Sango [13], and Agre and Rammaha [23], the main aim of this paper is to give several results concerning the local existence and the finite time blow-up of weak solutions of (1.1). By using the arguments in [13,23], we prove that under suitable assumptions on σ, p_i, and with certain initial data, the solutions of (1.1) exist locally and blow up in finite time.

The rest of our paper is organized as follows. In Section 2, we give some notations and state our main results. Section 3 is devoted to proving our main results.

2. Notation and main results

Consider the anisotropic Sobolev space

$$W^1_{0,\tilde{p}}(\Omega) = \left\{u \in W^{1,1}(\Omega) : u|_{\partial\Omega} = 0,\ \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega),\ i = 1,\ldots,n\right\},$$

with the norm

$$\|u\|_{W^1_{0,\tilde{p}}(\Omega)} = \sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i},$$

where $\tilde{p} = (p_1,\ldots,p_n)$. We denote the dual of $W^1_{0,\tilde{p}}(\Omega)$ by $(W^1_{0,\tilde{p}}(\Omega))'$.

Now we set

$$\check{p} = \min\{p_1,\ldots,p_n\},$$
$$\hat{p} = \max\{p_1,\ldots,p_n\},$$
$$\check{p}^* = \frac{n\check{p}}{n-\check{p}} \quad \text{if } \check{p} < n.$$

For the parameter σ, we make the following assumptions

$$\begin{cases} 1 < \sigma < \check{p}^*, & n > \check{p} \\ 1 < \sigma < +\infty, & n \leq \check{p}. \end{cases} \tag{2.1}$$

We are now in a position to state our main results. Our first theorem establishes the existence of a local weak solution to (1.1).

Theorem 2.1. *Assume (2.1) holds, the parameters σ, p_i satisfy*

$$\frac{\hat{p}}{2} + 1 < \sigma < \frac{\check{p}^*}{2} + 1, \tag{2.2}$$

and further the initial data u_0, u_1 satisfy the conditions

$$u_0 \in W^1_{0,\tilde{p}}(\Omega), \qquad u_1 \in L^2(\Omega).$$

Then there exists a weak solution $u(x,t)$ to (1.1), such that

$$u \in L^\infty(0,T; W^1_{0,\tilde{p}}(\Omega)); \qquad u_t \in L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; L^2(\Omega)),$$

for some $T > 0$.

Remark 1. When we take $p_i = 2$, (2.2) becomes $2 < \sigma < \frac{2n-2}{n-2}$, this condition coincides with some results in [17]; when we take $p_i = p > 2$, (2.2) becomes $\frac{p}{2} + 1 < \sigma < \frac{np}{2(n-p)} + 1$, which partially covers some results in [10].