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Sustainable dynamics of size-structured forest under climate change

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ABSTRACT

This work investigates the impact of global climate change on the sustainable growth of forest, namely, on its aggregated characteristics such as the number of trees, the basal area, and the amount of carbon sequestrated in the stand. The forest dynamics is described by a nonlinear size-structured population model. The existence of a steady state regime is proven and explicit formulas for the aggregated characteristics are obtained. A numeric simulation on realistic data illustrates and extends the analytic results obtained.

1. Introduction

Size-structured models of forest dynamics can be described using PDEs of a special type with nonlocal nonlinearities. This work follows a realistic forest model [1,2] that includes the size-dependent mortality and growth, intra-species competition, carbon sequestration in biomass, and other characteristics of the stand. The analysis focuses on effects of climate change. By [3–5], the climate change will primarily augment the growth rate of the stand, whereas its effect on the mortality cannot be determined precisely. Therefore, we compare qualitative dynamics of the forest for different growth rates related to various climate change scenarios. To examine sustainable forest management, we consider the infinite time horizon $[0, \infty)$. We analyze separately a managed forest with planted trees and a wild forest with natural reproduction of trees and establish a link between them.

A. *The model of a managed (controlled) forest* [1,2] is described by the following PDE:

$$\frac{\partial x(t,l)}{\partial t} + \frac{\partial[g(E(t),l)x(t,l)]}{\partial l} = -\mu(E(t),l)x(t,l), \quad t \in [0,\infty),\ l \in [l_0, l_m], \tag{1}$$

$$E(t) = \chi \int_{l_0}^{l_m} l^2 x(t,l)dl, \tag{2}$$

with boundary conditions

$$g(E(t),l_0)x(t,l_0) = p(t), \quad t \in [0,\infty), \qquad x(0,l) = x_0(l), \quad l \in [l_0, l_m], \tag{3}$$

where the tree size l is the tree diameter at breast height, $0 \le l_0 \le l \le l_m$. The given growth function $g(E(t), l)$ describes the change in the tree diameter over time, and the instantaneous mortality rate $\mu(E(t), l)$ is the probability of death of an

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l-sized tree at time t. Their dependence on E reflects the intensity of intra-species competition. The *unknown* variables in the model (1)–(3) are the forest density $x(t, l)$ and the basal area $E(t)$ of the entire stand. The integral $\int_{l_1}^{l_2} x(t, l)dl$ determines the number of trees with sizes between l_1 and l_2 at time t.

The model (1)–(3) describes a managed forest without natural reproduction, in which all trees of the diameter l_0 are planted. The boundary condition (3) relates the density $x(t, l_0)$ of young trees to the flux of planted trees $p(t)$.

B. *The model of a wild forest* with natural reproduction includes (1)–(3) and the additional *fertility equation*

$$p(t) = \int_{l_0}^{l_m} \alpha(E(t), t, l)x(t, l)dl, \tag{4}$$

in which the given function $\alpha(E, t, l)$ is the *size-specific fertility rate* [5]. In this model, the flux of young trees $p(t)$ in (3) is determined by the total number of offspring of the stand that reach the size l_o.

2. Steady state analysis of the forest model

To analyze a sustainable forest, we look for a solution of the problem, which does not depend on the current time t:

$$x(t, l) = x(l), \qquad E(t) = E, \quad l \in [l_0, l_m],\ t \in [0, \infty). \tag{5}$$

A necessary condition for the existence of steady state solutions in the model of controlled forest (1)–(3) is

$$p(t) = p = \text{const}, \quad t \in [0, \infty). \tag{6}$$

Then, a possible steady state regime $x(l)$ of (1)–(3) is described by the integral–differential equation

$$\frac{d[g(E, l)x(l)]}{dl} = -\mu(E, l)x(l), \quad l \in [l_0, l_m],\ E = \chi \int_{l_0}^{l_m} l^2 x(l)dl,\ g(E, l_0)x(l_0) = p. \tag{7}$$

If we treat E as a known parameter, then the initial problem (7) has the exact solution

$$x(l) = \frac{p}{g(E, l)} e^{-\int_{l_0}^{l} \frac{\mu(E,\xi)}{g(E,\xi)} d\xi}, \quad l \in [l_0, l_m]. \tag{8}$$

Formula (8) also holds for the wild forest model (1)–(4) and leads to an important link between these two models.

Theorem 1 (*On Connections Between the Wild and Controlled Forest Models*)**.** *The model (1)–(4) of a wild forest can possess a steady state solution (8) only if the reproduction number of the forest*

$$R(E) = \int_{l_0}^{l_m} \frac{\alpha(E, l)}{g(E, l)} e^{-\int_{l_0}^{l} \frac{\mu(E,\xi)}{g(E,\xi)} d\xi} dl \tag{9}$$

equals 1. *Then the steady state solution of the model (1)–(4) is the same as for the model (1)–(3) at $p = g(E, l_0)x(l_0)$.*

Proof. Follows from substituting (8) into equality (4) and using the notation (9). □

Theorem 1 allows us to focus on the steady state analysis of the controlled forest (1)–(3) and expand the results obtained to the wild forest (1)–(4). Here and hereafter, we analyze (1)–(3) under condition (6).

Theorem 2 (*On the Existence of a Steady State*)**.** *If (6) holds and $\mu(E, l) \geq \mu_{\min} > 0, g(E, l) \leq g_{\max} < \infty$ for $0 < E < \infty, l_0 \leq l^*(t) \leq l_m$, then there exists a positive value E^* that satisfies (7). This value E^* is unique if $|\mu_E(E, l)| \ll \mu(E, l)$ and $|g_E(E, l)| \ll g(E, l)$.*[1] *The unique solution $x(l)$ of (7) is expressed by (8) for $E = E^*$.*

Proof. Substituting (8) into the second equation of (7), we obtain the following nonlinear equation:

$$F(E) = E - \chi \int_{l_0}^{l_m} \frac{pl^2}{g(E, l)} e^{-\int_{l_0}^{l} \frac{\mu(E,\xi)}{g(E,\xi)} d\xi} dl = 0,$$

for E^*. The continuous function $F(E) < 0$ at $E = 0$ and $F(E) > 0$ for large E, which proves the existence of at least one $E^* > 0$ such that $F(E) = 0$. The derivative $F'(E) > 0$ under the theorem conditions, and hence the value E^* is unique. □

[1] These conditions mean that the functions $\mu(E, l)$ and $g(E, l)$ are slowly changing functions of E (their dependence on E is weak). The notation f_x means $\partial f/\partial x$.