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On the (s, t)-Pell and (s, t)-Pell–Lucas sequences and their matrix representations

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ABSTRACT

In this paper, we first give new generalizations for (s, t)-Pell $\{p_n(s, t)\}_{n\in\mathbb{N}}$ and (s, t)-Pell Lucas $\{q_n(s, t)\}_{n\in\mathbb{N}}$ sequences for Pell and Pell–Lucas numbers. Considering these sequences, we define the matrix sequences which have elements of $\{p_n(s, t)\}_{n\in\mathbb{N}}$ and $\{q_n(s, t)\}_{n\in\mathbb{N}}$. Then we investigate their properties.

1. Introduction

Fibonacci, Lucas, Pell, and Pell–Lucas sequences have been discussed in many articles and books (see [1–4]). For $n > 1$, the well-known Fibonacci $\{F_n\}$, Lucas $\{L_n\}$, Pell $\{p_n\}$, and Pell–Lucas $\{q_n\}$ sequences are defined as $F_n = F_{n-1} + F_{n-2}$, $L_n = L_{n-1} + L_{n-2}$, $p_n = 2p_{n-1} + p_{n-2}$, and $q_n = 2q_{n-1} + q_{n-2}$, where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, $p_0 = 0$, $p_1 = 1$, and $q_0 = 2$, $q_1 = 2$. The closed-form expressions for the Fibonacci and Lucas numbers are $F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$ and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Also, for $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$, the Pell number and Pell–Lucas number are $p_n = \frac{a^n-b^n}{a-b}$ and $q_n = a^n + b^n$, where a and b are the roots of the equation $x^2 = 2x + 1$. Further details about Pell and Pell–Lucas numbers can be found in [5].

In [6], Kılıç gave the definition of generalized Pell (p, i)-numbers and then presented their generating matrix. He obtained relationships between the generalized Pell (p, i)-numbers and their sums and permanents of certain matrices. Also, he derived the generalized Binet formulas, sums, combinatorial representations. In [7,8], the authors defined a new matrix generalization of the Fibonacci and Lucas numbers, and using essentially a matrix approach they showed properties of these matrix sequences.

2. The (s, t)-Pell, (s, t)-Pell–Lucas sequences and their matrix sequences

In this section, we give definitions of the (s, t)-Pell and (s, t)-Pell–Lucas sequences and (s, t)-Pell and (s, t)-Pell–Lucas matrix sequences. We also investigate their properties.

Definition 1. For any real numbers s, t and $n \geqslant 2$, let $s^2 + t > 0$, $s > 0$ and $t \neq 0$. Then the (s, t)-Pell sequence $\{p_n(s, t)\}_{n\in\mathbb{N}}$ and the (s, t)-Pell–Lucas sequence $\{q_n(s, t)\}_{n\in\mathbb{N}}$ are defined respectively by

$$p_n(s, t) = 2sp_{n-1}(s, t) + tp_{n-2}(s, t), \tag{1}$$

$$q_n(s, t) = 2sq_{n-1}(s, t) + tq_{n-2}(s, t), \tag{2}$$

with initial conditions $p_0(s, t) = 0$, $p_1(s, t) = 1$ and $q_0(s, t) = 2$, $q_1(s, t) = 2s$.

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Thus one can obtain the characteristic equation of (1) and (2) in the form $x^2 = 2sx + t$; then the roots of the characteristic equation of (1) and (2) are $x_1 = s + \sqrt{s^2+t}$ and $x_2 = s - \sqrt{s^2+t}$. Note that $x_1 + x_2 = 2s$, $x_1 - x_2 = 2\sqrt{s^2+t}$ and $x_1 x_2 = -t$.

For some special values of s and t in (1), it is obvious that the following results hold.

- If $s = \frac{1}{2}$, $t = 1$, the classic Fibonacci sequence is obtained.
- If $s = t = 1$, the classic Pell sequence is obtained.
- If $s = \frac{1}{2}$, $t = 2$, the classic Jacobsthal sequence is obtained.
- If $s = \frac{3}{2}$, $t = -2$, the Mersenne sequence is obtained.

Also, for some special values of s and t in (2), it is obvious that the following results hold.

- If $s = \frac{1}{2}$, $t = 1$, the classic Lucas sequence is obtained.
- If $s = t = 1$, the classic Pell–Lucas sequence is obtained.
- If $s = \frac{1}{2}$, $t = 2$, the classic Jacobsthal–Lucas sequence is obtained.
- If $s = \frac{3}{2}$, $t = -2$, the Fermat sequence is obtained.

Let us consider the following proposition, which will be needed for the results in this section. In fact, by this proposition, there will be given a relationship between the sequences $\{p_n(s,t)\}_{n\in\mathbb{N}}$ and $\{q_n(s,t)\}_{n\in\mathbb{N}}$.

Proposition 2. *For $n \geqslant 0$, we have*

$$q_n(s,t) = 2sp_n(s,t) + 2tp_{n-1}(s,t). \tag{3}$$

Proof. To prove the existence of this equality, we need to consider the sequence given in (2) with its initial conditions.

If we consider the initial condition $q_0(s,t) = 2$, then the expression can be written as

$$q_0(s,t) = 2 = (2s)\,0 + (2t)\,\frac{1}{t}.$$

If we apply same idea to the other condition $q_1(s,t) = 2s$, then we have

$$q_1(s,t) = 2s = (2s)\,1 + (2t)\,0.$$

In fact, these rewritten conditions contain the initial conditions $p_{-1}(s,t)$, $p_0(s,t)$ and $p_1(s,t)$ of the (s,t)-Pell sequence. Therefore, by replacing these conditions by these new $q_0(s,t)$ and $q_1(s,t)$, we obtain

$$q_0(s,t) = 2 = (2s)\,p_0(s,t) + (2t)\,p_{-1}(s,t),$$
$$q_1(s,t) = 2s = (2s)\,p_1(s,t) + (2t)\,p_0(s,t).$$

By keeping the (s,t)-Pell sequence and using same technique, we get

$$(2s)\,p_2(s,t) + (2t)\,p_1(s,t),$$

which gives $q_2(s,t)$ in the statement of proposition. So, by iterating process, we obtain the general term in the form $2sp_n(s,t) + 2tp_{n-1}(s,t)$, which implies $q_n(s,t)$, as required. □

In the following proposition, using the same approximation as in Proposition 2, we will show that there are also some other relations between $\{p_n(s,t)\}_{n\in\mathbb{N}}$ and $\{q_n(s,t)\}_{n\in\mathbb{N}}$ without any proof.

Proposition 3. *For $n \geqslant 0$, we have*

- $q_{n+2}^2(s,t) + tq_{n+1}^2(s,t) = 4(s^2+t)p_{2n+3}(s,t)$,
- $q_{n+2}^2(s,t) + tq_{n+1}^2(s,t) = q_{2n+4}(s,t) + tq_{2n+2}(s,t)$,
- $q_{2n}(s,t) = p_n(s,t)q_{n+1}(s,t) + tp_{n-1}(s,t)q_n(s,t)$.

Now, considering these sequences, we define the matrix sequences which have elements of (s,t)-Pell and (s,t)-Pell–Lucas sequences.

Definition 4. Let $s, t \in \mathbb{R}$, $s > 0$, $t \neq 0$, $s^2 + t > 0$, and $n \geqslant 2$. The (s,t)-Pell matrix sequence $\{\mathcal{P}_n(s,t)\}_{n\in\mathbb{N}}$ and (s,t)-Pell–Lucas matrix sequence $\{\mathcal{Q}_n(s,t)\}_{n\in\mathbb{N}}$ are defined respectively by

$$\mathcal{P}_n(s,t) = 2s\mathcal{P}_{n-1}(s,t) + t\mathcal{P}_{n-2}(s,t), \tag{4}$$
$$\mathcal{Q}_n(s,t) = 2s\mathcal{Q}_{n-1}(s,t) + t\mathcal{Q}_{n-2}(s,t), \tag{5}$$

with initial conditions $\mathcal{P}_0(s,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathcal{P}_1(s,t) = \begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix}$, and $\mathcal{Q}_0(s,t) = \begin{pmatrix} 2s & 2 \\ 2t & -2s \end{pmatrix}$, $\mathcal{Q}_1(s,t) = \begin{pmatrix} 4s^2+2t & 2s \\ 2st & 2t \end{pmatrix}$.

In the rest of this paper, the (s,t)-Pell and (s,t)-Pell–Lucas matrix sequences will be denoted by $\mathcal{P}_n$ and $\mathcal{Q}_n$ instead of $\mathcal{P}_n(s,t)$ and $\mathcal{Q}_n(s,t)$, respectively.