Applied Mathematics Letters 25 (2012) 1560–1564

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Efficient open domination in Cayley graphs

T. Tamizh Chelvam [a,*], Sivagnanam Mutharasu [b]

[a] Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli-627 012, India

[b] Department of Science and Humanities, Sri Krishna College of Engineering and Technology, Coimbatore-641 008, India

ARTICLE INFO

Article history:
Received 23 November 2010
Received in revised form 25 November 2011
Accepted 20 December 2011

Keywords:
Cayley graphs
Harary graphs
Efficient open domination
Covering projection

ABSTRACT

Efficient open dominating sets in bipartite Cayley graphs are characterized in terms of covering projections. Necessary and sufficient conditions for the existence of efficient open dominating sets in certain circulant Harary graphs are given. Chains of efficient dominating sets, and of efficient open dominating sets, in families of circulant graphs are described as an application.

1. Introduction

The terminology and notation in this paper follows that found in [1]. Cayley graphs are excellent models for interconnection networks and hence much investigated in connection with parallel processing and distributed computing. The concept of domination for Cayley graphs has been studied by various authors and one can refer to [2–5]. Dejter and Serra [2], and Huang and Xu [4] obtained some results on efficient dominating sets for Cayley graphs, whereas Obradovič et al. [5] studied efficient dominating sets in circulant graphs with two chord lengths. The existence of an independent perfect dominating set in a Cayley graph on an abelian group has been studied by Lee [3]. Tamizh et al. [6–8] studied some domination parameters such as domination, independent domination, connected domination and total domination for some classes of circulant graphs.

Let $G = (V, E)$ be a simple graph. For a vertex $v \in V(G)$, the *degree, open neighborhood* and *closed neighborhood* are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. A set $S \subseteq V$ is called a *dominating set* if every vertex $v \in V - S$ is adjacent to a vertex $u \in S$. The *domination number* $\gamma(G)$ of G equals the minimum cardinality among all dominating sets in G [1] and a corresponding dominating set is called a γ-set. A set $S \subseteq V$ is called a *total dominating* set if every vertex $v \in V$ is adjacent to a vertex $u(\neq v) \in S$. The *total domination number* $\gamma_t(G)$ equals the minimum cardinality among all total dominating sets in G [1] and a corresponding total dominating set is called a γ_t-set. Note that, if G is a k-regular graph, then a vertex can dominate k vertices and hence $\gamma_t(G) \geq \frac{|V(G)|}{k}$.

A dominating set S is said to be a *perfect* if $|N(v) \cap S| = 1$ for every vertex $v \in V - S$ [2]. A dominating set S is said to be *independent* if no two vertices of S are adjacent. A dominating set, which is both perfect and independent, is called an *efficient dominating set* [2]. Equivalently, a set $S \subseteq V$ is an efficient dominating set if for every vertex $v \in V$, $|N[v] \cap S| = 1$. A set $S \subseteq V$ is an *efficient open dominating set* if for every vertex $v \in V$, $|N(v) \cap S| = 1$ [1]. A *paired dominating set* S is a special case of a total dominating set in which the subgraph induced by S is a matching [9]. An efficient open dominating set is a special case of a paired dominating set. Note that, if S is an efficient open dominating set, then $|S|$ is even and we can

* Corresponding author.
E-mail addresses: tamche59@gmail.com (T. Tamizh Chelvam), skannanmunna@yahoo.com (S. Mutharasu).

0893-9659/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2011.12.036

write $S = A \cup B$ with $|A| = |B| = \frac{|S|}{2}$ such that every edge of the induced subgraph $\langle S\rangle$ has one end in A and another end in B.

Let Γ be a group with e as the identity element of Γ and X be a *generating set* of Γ such that $e \notin X$ and $X = X^{-1} = \{x^{-1}/x \in X\}$. A *Cayley graph* $G = Cay(\Gamma, X)$ is a graph with $V(G) = \Gamma$ and $E(G) = \{(x, xa)/x \in V(G), a \in X\}$. Since X is a generating set of Γ, $Cay(\Gamma, X)$ is a connected regular graph of degree $|X|$.

A graph $\tilde{G}$ is called a *covering graph* of G with *covering projection* $f : \tilde{G} \to G$ if there is a surjection $f : V(\tilde{G}) \to V(G)$ such that $f|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(G)$ with $\tilde{v} \in f^{-1}(v)$. A covering projection $f : \tilde{G} \to G$ is said to be a *m-fold covering projection* if f is m-to-one.

We will show that a bipartite Cayley graph G has an efficient open dominating set if and only if it is a covering graph of $K_{n,n}$ for some integer $n \geq 1$. We characterize some classes of circulant Harary graphs which admit efficient dominating sets. Finally, we construct chains of efficient dominating sets, and of efficient open dominating sets, in families of circulant graphs as an application of covering projection. Throughout this paper, $V(K_{n,n}) = (Y, Z)$, where Y=$\{y_1, y_2, \ldots, y_n\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$. We list below Theorems 1–3. Theorem 1 gives a necessary and sufficient condition for the existence of an efficient dominating set in a Cayley graph. In Section 2, we generalize the theorem for the existence of an efficient open dominating set in a bipartite Cayley graph and use Theorems 2 and 3 in what follows. In our results, we employ proof techniques used in [3] with necessary modifications.

Theorem 1 (*Lee [3]*). *Let $p : \tilde{G} \to G$ be a covering projection and let S be a perfect dominating set of G. Then $p^{-1}(S)$ is a perfect dominating set of $\tilde{G}$. Moreover, if S is independent, then $p^{-1}(S)$ is independent.*

Theorem 2 (*Haynes et al. [1]*). *If G has an efficient open dominating set S, then $|S| = \gamma_t(G)$ and all efficient open dominating sets have the same cardinality.*

Theorem 3 (*Jia Huang and Jun-Ming Xu [4]*). *Let G be a k-regular graph. Then $\gamma(G) \geq \frac{n(G)}{k+1}$, with the equality if and only if G has an efficient dominating set. In addition, if G has an efficient dominating set, then every efficient dominating set must be a γ-set, and vice versa.*

2. Efficient open dominating sets in bipartite Cayley graphs

In this section, we characterize bipartite Cayley graphs which admit efficient open dominating sets.

Lemma 4. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a generating set of a group Γ and S be an efficient open dominating set of $G = Cay(\Gamma, X)$. Then we have the following:*

(a). *For each i with $1 \leq i \leq n$, x_iS is an efficient open dominating set in G.*
(b). *$\{Sx_1, Sx_2, \ldots, Sx_n\}$ is a vertex partition of G.*

Proof. (a) Note that, for each $v \in V(G)$, we have an automorphism defined by $f_v(w) = vw$ for all $w \in V(G)$. Since the automorphic image of an efficient open dominating set is also an efficient open dominating set, x_iS is an efficient open dominating set in G for $1 \leq i \leq n$.

(b) Since S is a total dominating set, every $v \in V(G)$ is adjacent to some vertex $s \in S$ and so $v = sx_i$ for some $s \in S$ and $x_i \in X$. This means that $V(G) = Sx_1 \cup Sx_2 \cup \cdots \cup Sx_n$. Since G is n-regular and S is an efficient open dominating set, $|S| = \frac{|V(G)|}{n}$ and so $|V(G)| = n|S|$. Since $|S| = |Sx_1| = |Sx_2| = \cdots = |Sx_n|$, $\{Sx_1, Sx_2, \ldots, Sx_n\}$ is a vertex partition of G. $\square$

Lemma 5. *Let $S_1, S_2, \ldots, S_n$ be pairwise disjoint efficient open dominating sets of a graph G and $\tilde{G}$ be the subgraph of G, induced by $S_1 \cup S_2 \cup \cdots \cup S_n$. Let $m = \frac{|S_1|}{2}$. If $\tilde{G}$ is bipartite, then there exists a m-fold covering projection from $\tilde{G}$ onto $K_{n,n}$.*

Proof. As $\tilde{G}$ is bipartite, let $\tilde{G} = (M, N)$ be a bipartition of $V(\tilde{G})$. Note that each edge of $\langle S_i\rangle$ has one end in M and the other end in N. Let $M_i = S_i \cap M$ and $N_i = S_i \cap N$ for $1 \leq i \leq n$. Since each S_i is an efficient open dominating set in $\tilde{G}$, each vertex of M_i is adjacent with exactly one vertex of N_j for all j with $1 \leq j \leq n$ and each vertex of N_i is adjacent with exactly one vertex of M_j for all j with $1 \leq j \leq n$. Let $V(K_{n,n}) = (Y, Z)$, where $Y = \{y_1, y_2, \ldots, y_n\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$.

Define $f : \tilde{G} \to K_{n,n}$ by $f(x) = y_i$ if $x \in M_i$ and $f(x) = z_i$ if $x \in N_i$. Let $v \in V(K_{n,n})$. If $v = z_i$, then $f^{-1}(v) = N_i$ and $N(v) = \{y_1, y_2, \ldots, y_n\}$. Let $\tilde{v} \in f^{-1}(v) = N_i$. Since each vertex of N_i is adjacent with exactly one vertex of M_j for all j with $1 \leq j \leq n$, it follows that $N(\tilde{v}) = \{\beta_1, \beta_2, \ldots, \beta_n\}$, where $\beta_j \in M_j$ for $1 \leq j \leq n$. By the definition of f, $f(\beta_j) = y_j$, for $1 \leq j \leq n$. From this, one can conclude that $f|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection, when $v = z_i$. A similar proof follows for the case $v = y_i$. Since $m = \frac{|S_1|}{2} = |M_1| = |N_1|$, by Theorem 2, we have $\frac{|S_i|}{2} = |M_i| = |N_i| = m$ for each i with $1 \leq i \leq n$ and so f is an m-fold covering projection from $\tilde{G}$ onto $K_{n,n}$. $\square$

From Lemmas 4 and 5, we have the following corollary.