Applied Mathematics Letters 24 (2011) 224–228

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Approximation of fixed points of pseudocontraction semigroups based on a viscosity iterative process

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ARTICLE INFO

Article history:
Received 26 May 2010
Accepted 12 September 2010

Keywords:
Common fixed point
Nonexpansive mapping
Pseudocontraction semigroup
Variational inequality

ABSTRACT

In this paper, Moudafi's viscosity approximations with continuous strong pseudocontractions for a pseudocontraction semigroup are considered. A strong convergence theorem of fixed points is established in the framework of Banach spaces.

1. Introduction and preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\,\|x^*\|, \|x\| = \|x^*\|\}, \quad \forall x \in C,$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between E and E^*. In what follows, we denote a single valued normalized duality mapping by j.

Let C be a nonempty closed convex subset of a Banach space E and T a nonlinear mapping. From now on, we use $F(T)$ to denote the fixed point set of T.

Recall that T is said to be pseudocontractive if there exists some $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2, \quad \forall x, y \in C.$$

T is said to be strongly pseudocontractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \alpha\|x-y\|^2, \quad \forall x, y \in C$$

for some $j(x-y) \in J(x-y)$. T is said to be Lipschitz if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L\|x-y\|, \quad \forall x, y \in C.$$

If $L = 1$, then T is said to be nonexpansive.

The class of pseudocontractions is one of the most important classes of mappings among nonlinear mappings. Within the past 40 years or so, many authors have been devoted to the studies on the existence and convergence of fixed points for pseudocontractions.

In 1974, Deimling [1] proved the following existence result for continuous strong pseudocontractions in Banach spaces.

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doi:10.1016/j.aml.2010.09.008

Theorem D. *Let E be a Banach space, C be a nonempty closed convex subset of E and $T : C \to C$ be a continuous and strong pseudocontraction. Then T has a unique fixed point in C.*

A pseudocontraction semigroup is a family $\mathcal{F} = \{T(t) : t \geq 0\}$ of self-mappings of C such that

(a) $T(0)x = x$ for all $x \in C$;
(b) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
(c) $\lim_{t\to 0^+} T(t)x = x$ for all $x \in C$;
(d) for each $t > 0$, $T(t)$ is pseudocontractive; that is,

$$\langle T(t)x - T(t)y, j(x-y)\rangle \leq \|x-y\|^2, \quad \forall x, y \in C.$$

In this paper, we use Ω to denote the set of common fixed points of $\mathcal{F}$; that is,

$$\Omega := \{x \in C : T(t)x = x, t > 0\} = \bigcap_{t>0} F(T(t)).$$

Note that the class of pseudocontractive semigroups includes the class of nonexpansive semigroups as a special case.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [2–4]. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad \forall x \in C,$$

where $u \in C$ is a fixed point. Banach's Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [2] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T that is nearest to u. Reich [3] extended Broweder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

It is an interesting problem to extend Browder's and Reich's results to the contraction semigroup case. In 2003, Suzuki [5] proved the following results.

Theorem S. *Let C be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on C such that $\bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Let α_n and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad \forall n \geq 1. \tag{1.1}$$

Then $\{x_n\}$ converges strongly to the element of F nearest to u.

In 2005, Xu [6] extended Theorem S from Hilbert spaces to Banach spaces. To be more precise, he proved the following theorem.

Theorem X. *Let E be a uniformly convex Banach space having a weakly continuous duality map J_φ with gauge φ, C a nonempty closed convex subset of E and $\{T(t) : t \geq 0\}$ a nonexpansive semigroup on C such that $\Omega \neq \emptyset$. If $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$, then $\{x_n\}$ generated in (1.1) converges strongly to a member of Ω.*

Moudafi's viscosity approximation methods have been recently studied by many authors; see the well known results in [7,8]. However, the involved mapping f is usually considered as a contraction. Note that Suzuki [9] proved the equivalence between Moudafi's viscosity approximation with contractions and Browder-type iterative processes (Halpern-type iterative processes); see [9] for more details.

The purpose of this paper is to consider a pseudocontraction semigroup based on Moudafi's viscosity approximation with continuous strong pseudocontractions in the framework of Banach spaces. The results presented in this paper mainly improved and extended the corresponding results announced in [10,11,5,12,8,6].

In order to prove our main result, we need the following lemmas and definitions.

Let l^∞ be the Banach space of all bounded real-valued sequences. A Banach limit LIM is a linear continuous functional on l^∞ such that

$$\|\mathrm{LIM}\| = \mathrm{LIM}(1) = 1, \qquad \mathrm{LIM}(t_1, t_2, \dots) = \mathrm{LIM}(t_2, t_3, \dots)$$

for each $t = (t_1, t_2, \dots) \in l^\infty$. If LIM is a Banach limit, then

$$\liminf_{n\to\infty} t_n \leq \mathrm{LIM}(t) \leq \limsup_{n\to\infty} t_n, \quad \forall t = (t_1, t_2, \dots) \in l^\infty.$$