Applied Mathematics Letters 24 (2011) 27–32

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On a subclass of strongly starlike functions

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ARTICLE INFO

Article history:
Received 4 November 2009
Received in revised form 20 July 2010
Accepted 1 August 2010

Keywords:
Analytic functions
Convex functions
Starlike functions
Strongly starlike functions
Radius of starlikeness of order α
Radius of convexity of order α

ABSTRACT

Let $\mathcal{S}^*(q_c)$, $c \in (0, 1]$, denote the class of analytic functions f in the unit disc $\mathcal{U}$ normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition

$$\left|\left[zf'(z)/f(z)\right]^2 - 1\right| |< c, \quad z \in \mathcal{U}.$$

The relations between $\mathcal{S}^*(q_c)$ and other classes geometrically defined are considered. The radii of convexity (starlikeness) of order α are calculated. The same problem in the class of strongly starlike functions of order β is also considered.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathcal{U}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0, f'(0) = 1$. Robertson introduced in [1] the classes $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ of starlike and convex functions of order $\alpha \leq 1$, which are defined by

$$\mathcal{S}^*(\alpha) := \left\{f \in \mathcal{A} : \mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha,\ z \in \mathcal{U}\right\}, \qquad \mathcal{K}(\alpha) := \left\{f \in \mathcal{A} : zf'(z) \in \mathcal{S}^*(\alpha)\right\}. \tag{1}$$

If $\alpha \in [0; 1)$, then a function in either of these sets is univalent, if $\alpha < 0$ it may fail to be univalent. In particular $\mathcal{S}^*(0) = \mathcal{S}^*$ is the class of starlike functions, while $\mathcal{K}(0) = \mathcal{K}$ is the class of convex functions. One can alter the conditions (1) to define a new geometric property of functions. In this way many interesting classes of analytic functions have been defined (see for instance [2]). Let $c \in (0, 1]$ be given and let us consider the class $\mathcal{S}^*(q_c)$:

$$\mathcal{S}^*(q_c) = \left\{f \in \mathcal{A} : \left|\left[\frac{zf'(z)}{f(z)}\right]^2 - 1\right| < c,\ z \in \mathcal{U}\right\}. \tag{2}$$

Some results about this class were given in [3]. The class $\mathcal{S}^*(q_1) = \mathcal{SL}^*$ was considered in [4]. We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in $\mathcal{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$ and $f(z) = g(\omega(z))$. In particular, if g is univalent in $\mathcal{U}$, then we have the following equivalence

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(|z| < 1) \subset g(|z| < 1). \tag{3}$$

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doi:10.1016/j.aml.2010.08.004

It is easy to see that $f \in \mathcal{S}^*(q_c)$, $c \in (0,1]$ if and only if it satisfies the differential subordination

$$zf'(z)/f(z) \prec q_c(z) = \sqrt{1+cz} \quad z \in \mathcal{U}, \tag{4}$$

where the branch of the square root is chosen in order to $\sqrt{1} = 1$, and $c \in (0, 1]$. Let us denote by $\mathcal{O}_c$ a set of all points on the right half-plane such that the product of the distances from each point to the focuses -1 and 1 is less then c:

$$\mathcal{O}_c := \{w \in \mathbb{C} : \mathfrak{Re}\, w > 0,\ |w^2 - 1| < c\},$$

therefore its boundary $\partial\mathcal{O}_c$ is the right loop of the Cassinian Ovals

$$(x^2+y^2)^2 - 2(x^2-y^2) = c^2 - 1 \tag{5}$$

which becomes the right half of the Lemniscate of Bernoulli when $c = 1$. In the polar coordinates the Eq. (5) becomes

$$r^4 - 2r^2 \cos 2\theta = c^2 - 1. \tag{6}$$

2. Main results

Theorem 1. *Let $c \in (0, 1]$. Then we have*

(i) $\left\{w : |w-1| < \sqrt{1+c} - 1\right\} \subset \mathcal{O}_c$,
(ii) $\mathcal{O}_c \subset \left\{w : \left|w - \frac{\sqrt{1+c}+\sqrt{1-c}}{2}\right| < \frac{\sqrt{1+c}-\sqrt{1-c}}{2}\right\}$,
(iii) $\mathcal{O}_c \subset \left\{w : \sqrt{1-c} < \mathfrak{Re} w < \sqrt{1+c}\right\}$, $\mathcal{O}_c \subset \left\{w : |\mathfrak{Im} w| < \frac{c}{2}\right\}$,
(iv) $\mathcal{O}_c \subset \left\{w : |\mathrm{Arg} w| < \frac{1}{2} \arccos \sqrt{1-c^2}\right\}$.

Proof. It is easy to see that if $|w-1| < \sqrt{1+c} - 1$, then $|w+1| < \sqrt{1+c} + 1$. Multiplying these inequalities we obtain $|w^2 - 1| < c$ so $w \in \mathcal{O}_c$ and (i) is proved. For the proof of (ii) let us suppose that $w \in \mathcal{O}_c$, $w = x + iy$, $x > 0$. Then by (5) we have

$$(x^2+y^2)^2 - 2(x^2-y^2) < c^2 - 1, \quad x > 0 \tag{7}$$

and so

$$(x^2+y^2)^2 - 2\left(x^2 - y^2\sqrt{1-c^2}\right) < c^2 - 1, \quad x > 0. \tag{8}$$

After some calculation we can obtain the following equivalent form of the inequality (8)

$$\left[x^2 + y^2 + \sqrt{1-c^2}\right]^2 < \left[x(\sqrt{1-c} + \sqrt{1+c})\right]^2, \quad x > 0,$$

thus

$$x^2 + y^2 + \sqrt{1-c^2} < x(\sqrt{1-c} + \sqrt{1+c}), \quad x > 0. \tag{9}$$

Combining with (9) we conclude that

$$\left[x - (\sqrt{1+c} + \sqrt{1-c})/2\right]^2 + y^2 < \left[(\sqrt{1+c} - \sqrt{1-c})/2\right]^2.$$

This proves (ii). The conditions (iii) become clear by looking at Fig. 1. For the proof of (iv) let us consider the equation of the boundary $\partial\mathcal{O}_c$ in the polar coordinates (6). Then we have

$$\cos 2\theta = (r^4 + 1 - c^2)/2r^2 =: S(r), \tag{10}$$

and $S_{\min}(\sqrt[4]{1-c^2}) = \sqrt{1-c^2}$. Thus from (10) we obtain (iv) which completes the proof of Theorem 1. □

Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β

$$\mathcal{SS}^*(\beta) := \left\{f \in \mathcal{A} : \left|\mathrm{Arg}\frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2},\ z \in \mathcal{U}\right\}, \quad 0 < \beta \le 1 \tag{11}$$

which was introduced in [5,6]. The class $\mathcal{S}^*[A, B]$

$$\mathcal{S}^*[A,B] := \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},\ z \in \mathcal{U}\right\} \tag{12}$$

was investigated in [7] for $-1 \le B < A \le 1$. The next corollary expresses Theorem 1 in terms of the above classes.