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A note on sums of products of Bernoulli numbers

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ABSTRACT

In this work we obtain a new approach to closed expressions for sums of products of Bernoulli numbers by using the relation of values at non-positive integers of the important representation of the multiple Hurwitz zeta function in terms of the Hurwitz zeta function.

1. Introduction and preliminaries

Let n be a positive integer and $x \in \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$.
Let $\zeta_n(s, x)$ denote the multiple Hurwitz zeta function defined by

$$\zeta_n(s, x) = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{1}{(x + k_1 + \cdots + k_n)^s}, \quad \mathrm{Re}\,(s) > n \tag{1.1}$$

and for $\mathrm{Re}\,(s) \leq n, s \neq 1, 2, \ldots, n$, by the analytic continuation. Then, in terms of the familiar higher-order Bernoulli polynomials $B_k^{(n)}(x)$ defined by means of the generating function

$$\left(\frac{t}{e^t - 1}\right)^n e^{xt} = \sum_{k=0}^{\infty} B_k^{(n)}(x) \frac{t^k}{k!}, \quad n \geq 1 \tag{1.2}$$

it is known that

$$\zeta_n(-k, x) = (-1)^n \frac{k!}{(k+n)!} B_{k+n}^{(n)}(x) \tag{1.3}$$

for $k = 0, 1, \ldots$ (cf. [1–3]). In particular we have $B_k^{(n)}(0) = B_k^{(n)}$, the higher-order Bernoulli numbers. When $n = 1$, $\zeta_1(s, x) = \zeta(x, s)$ is the well-known Hurwitz zeta function. The multiple Hurwitz zeta function $\zeta_n(s, x)$ at positive integers which are greater than or equal to n is closely related the multiple gamma functions as an extension of the classical Euler gamma functions $\Gamma(x)$ (to be introduced below), the so-called Barnes multiple gamma functions $\Gamma_n(x)$ with the parameter x is defined by $\Gamma_n(x) = \exp(\frac{\partial}{\partial s}\zeta_n(s, x)|_{s=0}) = \prod_{k_1,\ldots,k_n=0}^{\infty}(x+k_1+\cdots+k_n)^{-1}$ (cf. [3]). The multiple gamma function, originally introduced over 100 years ago, has significant applications in connection with the Riemann Hypothesis (cf. [3,4]).

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From (1.2), we easily obtain

$$B_k^{(n+1)} = \left(1 - \frac{n}{k}\right) B_k^{(n)} - kB_{k-1}^{(n)}; \tag{1.4}$$

$$B_k^{(n)} = -\frac{k}{n} \sum_{i=1}^{k} (-1)^i \binom{k}{i} B_i B_{k-i}^{(n)}; \tag{1.5}$$

$$B_k^{(n)}(x) = (-1)^k B_k^{(n)}(n-x); \tag{1.6}$$

$$B_k^{(n)}(x) = \sum_{\substack{k_1,\dots,k_n \geq 0 \\ k_1+\cdots+k_n=k}} \binom{k}{k_1,\dots,k_n} B_{k_1}(x_1)\cdots B_{k_n}(x_n), \tag{1.7}$$

where $x = x_1 + \cdots + x_n$.

Set

$$x(x+1)\cdots(x+n-1) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} x^j, \tag{1.8}$$

where $\begin{bmatrix} n \\ j \end{bmatrix}$ are the Stirling cycle numbers, defined recursively by

$$\begin{bmatrix} n \\ j \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ j \end{bmatrix} + \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}, \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Then

$$\binom{k+n-1}{n-1} = \sum_{l=0}^{n-1} p_{n,l}(x+n)(x+n+k)^l,$$

where $p_{n,l}(x+n)$ is a polynomial in x defined by (see, for details, [3])

$$p_{n,l}(x+n) = \frac{1}{(n-1)!} \sum_{j=l}^{n-1} (-1)^{j-l}(x+n)^{j-l} \binom{j}{l} \begin{bmatrix} n \\ j+1 \end{bmatrix}. \tag{1.9}$$

By the above equations, the multiple Hurwitz zeta function defined by (1.1) may be expressed in terms of the Hurwitz zeta function

$$\zeta_n(s, x+n) = \sum_{l=0}^{n-1} p_{n,l}(x+n)\zeta(s-l, x+n), \quad x \geq 0, \tag{1.10}$$

which is due to Mellin [5], Choi [1], Vardi [4] and Kanemitsu et al. [2].

A well-known relation among the Bernoulli numbers is (for $n \geq 2$)

$$\sum_{i=1}^{k-1} \binom{2k}{2i} B_{2i}B_{2k-2i} = -(2k+1)B_{2k}, \quad k \geq 2. \tag{1.11}$$

This was found by many authors, including Euler; for references, see, e.g., [6–13].

Eie [14] and Sita Rama Chandra Rao and Davis [13] considered the sum of products of three and four Bernoulli numbers.

Dilcher [6] established the following interesting sums of products of Bernoulli numbers:

$$\sum_{\substack{r_1,\dots,r_n \geq 0 \\ r_1+\cdots+r_n=r}} \binom{2r}{2r_1,\dots,2r_n} B_{2r_1}\cdots B_{2r_n} = \begin{cases} \dfrac{(2r)!}{(2r-n)!} \displaystyle\sum_{k=0}^{\left[\frac{n-1}{2}\right]} b_k^{(n)} \frac{B_{2r-2k}}{2r-2k}, & 2r > n \\ \dfrac{(2r)!}{4^r} + \displaystyle\sum_{k=0}^{r-1} \frac{(2r)!}{2r-2k} b_k^{(2r)} B_{2r-2k}, & 2r = n \\ (-1)^{n-1}(n-2r-1)!(2r)!b_r^{(n)}, & 0 \leq r \leq \left[\dfrac{n-1}{2}\right], \end{cases} \tag{1.12}$$

where

$$\binom{2r}{2r_1,\dots,2r_n} = \frac{(2r)!}{(2r_1)!\cdots(2r_n)!}$$