Applied Mathematics Letters 24 (2011) 279–282

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

An iterative algorithm for solving a kind of discrete HJB equation with M-functions ☆

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ARTICLE INFO

Article history:
Received 5 June 2009
Accepted 3 October 2010

Keywords:
HJB equation
Nonlinear Gauss–Seidel method
Domain decomposition method
M-function

ABSTRACT

In this work, an iterative algorithm for solving a kind of discrete HJB equation with M-functions is proposed and monotone convergence is obtained. Furthermore, a domain decomposition method based on the iterative algorithm is also presented.

1. Introduction

The Hamilton–Jacobi–Bellman (HJB) equations have many applications in science, engineering and economics; see for example [1–3] and references therein. They can arise in solving optimal control problems by dynamic programming techniques. Many nonlinear option pricing problems can also be formulated as optimal control problems, leading to HJB equations.

This work is concerned with the following HJB equation:

$$\begin{aligned}&\text{find} \quad u \in R^n, \quad \text{such that}\\ &\max_{1\le j\le k}\{\mathcal{A}^j(u) - F^j\} = 0,\end{aligned} \tag{1.1}$$

where $F^j \in R^n$, $\mathcal{A}^j$ are M-functions, $j = 1, 2, \ldots, k$. Here, the M-function is defined as follows:

Definition 1.1 (*[4]*)**.** Let K be a closed subset of R^n. Function $\mathcal{A} : K \to R^n$ is called an M-function over K if it satisfies the following two conditions:

(1) inverse isotone: for any $u, v \in K$, if $\mathcal{A}(u) \ge \mathcal{A}(v)$, $u \ge v$;
(2) off-diagonal antitone: for any pair of indices i, j satisfying $i \ne j$ and any $v \in K$, the one-dimensional function $f_{ij}(t) : V_i \to R$ defined as

$$f_{ij}(t) \equiv \mathcal{A}_j(v_1, \ldots, v_{i-1}, t, v_{i+1}, \ldots, v_n)$$

is a nonincreasing function, where $V_i = \{t \in R : (v_1, \ldots, v_{i-1}, t, v_{i+1}, \ldots, v_n) \in K\}$.

☆ The work was supported by the National Natural Science Foundation of PR China grant 10971058.
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0893-9659/$ – see front matter
doi:10.1016/j.aml.2010.10.004

It is easy to check that the linear function $\mathcal{A}(v) = Av + b$ is an M-function if A is an M-matrix.

In the last few decades, many numerical schemes have been proposed for solving HJB equations; see for example [5–9] and references therein. Lions and Mercier [5] presented two iterative algorithms for solving HJB equations. At each iteration, a linear complementarity subproblem or a linear equation system subproblem is solved. Sun [7] gave a domain decomposition method based on the algorithms proposed by Lions and Mercier [5]. Camilli et al. [8] proposed a domain decomposition method for solving the discrete problem of a kind of HJB equation of first order. Zhou and Zou [9] presented a successive relaxation iterative algorithm for solving the discrete HJB equation. The advantage of the algorithm in [9] is that it does not need to solve a subproblem but carries out arithmetic operations at each iteration.

The numerical algorithms mentioned above concern the case where $\mathcal{A}^j$ are matrices, $j = 1, 2, \ldots, k$. Motivated by recent work going on in this field, we go on considering the numerical solution of the HJB equations. We propose an iterative algorithm for solving the HJB equation, which is similar to the nonlinear Gauss–Seidel algorithm for solving systems of nonlinear equations. We obtain the monotone convergence of the algorithm. It can be used as some smoother if a new nonlinear multigrid method for (1.1) is constructed. Furthermore, we present a domain decomposition method for solving (1.1). This kind of domain decomposition method based on the proposed iterative algorithm is an extension of the domain decomposition method for solving HJB equations with $\mathcal{A}^j$ being matrices, $j = 1, 2, \ldots, k$.

The work is organized as follows. In Section 2, we present an iterative algorithm for solving HJB equations, and discuss its monotone convergence. In Section 3, we propose a domain decomposition method for solving (1.1) and establish its convergence.

2. The iterative algorithm and its convergence

In this section, we present an iterative algorithm for solving (1.1) and discuss the monotone convergence of the algorithm. We call $v \in R^n$ a supersolution for (1.1) if

$$\max_{1\le j\le k}\{\mathcal{A}^j(v) - F^j\} \ge 0.$$

The set of all supersolutions for (1.1) is denoted by S.

Now, we are ready to present the iterative algorithm for solving (1.1).

Algorithm 2.1.

Step 1. $\varepsilon > 0$, $u^0 \in S$, $m := 0$, $l := 1$ are given.

Step 2. For $l = 1, 2, \ldots, n$, compute u_l^{m+1}, such that

$$\max_{1\le j\le k}\{(\mathcal{A}^j(u^{m+1,l}) - F^j)_l\} = 0,$$

where $u^{m+1,l} = (u_1^{m+1}, \ldots, u_l^{m+1}, u_{l+1}^m, \ldots, u_n^m)$.

Step 3. Let $u^{m+1} = u^{m+1,n}$. If $\|u^{m+1} - u^m\| < \varepsilon$, stop. Otherwise, let $m := m + 1$, $l := 1$; go to Step 2.

Throughout the work, we assume that Eq. (1.1) has a solution $u^\star$. Let $\mathcal{A}_i^{j_i}(u)$ denote the i-th component of $\mathcal{A}^{j_i}(u)$, $j_i = 1, 2, \ldots, k$, $i = 1, 2, \ldots, n$. The following assumption is essential.

Assumption 2.1. Functions in the set $\hat{\mathcal{A}}(u) = \{(\mathcal{A}_1^{j_1}(u), \mathcal{A}_2^{j_2}(u), \ldots, \mathcal{A}_n^{j_n}(u))^T, j_i = 1, 2, \ldots, k, i = 1, 2, \ldots, n\}$, are all M-functions.

Remark 2.1. Obviously, when $\mathcal{A}^j$, $j = 1, 2, \ldots, k$, arise from the discretization of elliptic operators of second order with nonlinear source terms, Assumption 2.1 holds.

In order to establish the theorem of convergence of Algorithm 2.1, we first give some useful lemmas.

Lemma 2.1 ([10]). *Let $\mathcal{A} : K \to R^n$ be an M-function, and I, J be a nonoverlapping decomposition of N. For any vectors $v, w \in K$, if $v_I \le w_I$ and $\mathcal{A}_J(v) \le \mathcal{A}_J(w)$, $v \le w$.*

Lemma 2.2. *Suppose that Assumption 2.1 holds. Then, the set S is bounded below.*

Proof. For any $u \in S$, we have $\max_{1\le j\le k}\{(\mathcal{A}^j(u) - F^j)_l\} \ge 0$, $l = 1, 2, \ldots, n$. Hence, for any $l \in \{1, 2, \ldots, n\}$, there exists a $j_l^\star \in \{1, 2, \ldots, k\}$, such that

$$(\mathcal{A}^{j_l^\star}(u) - F^{j_l^\star})_l = \max_{1\le j\le k}\{(\mathcal{A}^j(u) - F^j)_l\} \ge 0. \tag{2.1}$$

Since $u^\star$ is the solution of (1.1), we have

$$(\mathcal{A}^{j_l^\star}(u^\star) - F^{j_l^\star})_l \le \max_{1\le j\le k}\{(\mathcal{A}^j(u^\star) - F^j)_l\} = 0.$$