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Starlikeness criteria for a certain class of analytic functions☆

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ABSTRACT

We denote by $\mathcal{A}$, the class of all analytic functions f in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. For a positive number $\lambda > 0$, we denote by $\mathcal{U}_3(\lambda)$ the class of all $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, such that $a_3 - a_2^2 = 0$, and satisfying the condition

$$\left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, \quad z \in \Delta.$$

A function $f \in \mathcal{A}$ is said to be in $\mathcal{SR}(\gamma)$ if $|\arg f'(z)| < \pi\gamma/2$. In this paper, we find conditions on λ, α and γ such that $\mathcal{U}_3(\lambda)$ is included in the class of all starlike functions of order α, or the class of all strongly starlike functions of order γ, or $\mathcal{SR}(\gamma)$, respectively.

1. Introduction

We denote by $\mathcal{A}$ the class of all analytic functions f in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. For a positive number $\lambda > 0$, we set

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, z \in \Delta \right\}.$$

It is well known that [1] $\mathcal{U}(\lambda) \subsetneq \mathcal{S}$ for $0 < \lambda \le 1$, where $\mathcal{S}$ denotes the class of all univalent functions $f \in \mathcal{A}$. Several properties of $\mathcal{U}(\lambda)$ together with its generalization have been discussed in detail in recent papers; for example, see [2–5]. A function $f \in \mathcal{A}$ is called starlike if f is univalent and the image $f(\Delta)$ is starlike with respect to the origin. The class of all starlike functions is denoted by $\mathcal{S}^*$. It is well known that $f \in \mathcal{A}$ is starlike if and only if $\mathrm{Re}\,(zf'(z)/f(z)) > 0$. Furthermore, for a constant $\alpha \in [0, 1)$, $f \in \mathcal{A}$ is starlike of order α if $\mathrm{Re}\,(zf'(z)/f(z)) > \alpha$. We denote the class of all starlike functions of order α by $\mathcal{S}^*(\alpha)$. For a constant $\gamma \in (0, 1]$, a function $f \in \mathcal{A}$ is called strongly starlike of order γ if $|\arg(zf'(z)/f(z))| < \pi\gamma/2$. We denote by $\mathcal{SS}(\gamma)$ the set of all strongly starlike functions of order γ. As is well known, for $\gamma \in (0, 1)$, each function in $\mathcal{SS}(\gamma)$ is bounded. Note that, $\mathcal{SS}(\gamma) \subset \mathcal{SS}(1) \equiv \mathcal{S}^*(0) = \mathcal{S}^*$ for $\gamma \in (0, 1]$. Clearly, $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*$ for $\alpha \in [0, 1)$. These classes of functions were investigated by several authors, e.g. Sugawa [6–8].

We say that $f \in \mathcal{SR}(\gamma)$ if $f \in \mathcal{A}$ and $|\arg f'(z)| < \pi\gamma/2$. It is well known that $\mathcal{SR}(1) \equiv \mathcal{R}$ is included in $\mathcal{S}$. However, without an additional condition a function in $\mathcal{R}$ does not belong to $\mathcal{S}^*$.

A well known result of Fekete and Szegö (see [9]) shows that if $\mu \in [0, 1]$, then the inequality $|a_3 - \mu a_2^2| \le 1 + 2\exp(-2\mu/(1-\mu))$ holds for any $f \in \mathcal{S}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. In particular, for $\mu = 1$ one has $|a_3 - a_2^2| \le 1$ if $f \in \mathcal{S}$. Note that the quantity $a_3 - a_2^2$ represents $S_f(0)/6$, where S_f denotes the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$

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of locally univalent functions f in Δ. In the literature, there exists a large number of results about inequalities for $a_3-\mu a_2^2$ corresponding to various subclasses of $\mathcal{S}$.

In this paper, we consider the class of functions $f\in\mathcal{U}(\lambda)$ for which $a_3-a_2^2=0$. We call the class of all such functions by $\mathcal{U}_3(\lambda)$. In this paper, we find conditions on λ, α and γ such that $\mathcal{U}_3(\lambda)$ is included in $\mathcal{S}^*(\alpha)$ or $\mathcal{SS}(\gamma)$ or $\mathcal{SR}(\gamma)$.

Suppose that $f\in\mathcal{U}_3(\lambda)$. Then a simple calculation shows that

$$-z\left(\frac{z}{f(z)}\right)'+\frac{z}{f(z)}=\left(\frac{z}{f(z)}\right)^2 f'(z)=1+A_3z^3+\cdots:=1+\lambda w(z),\quad w\in\mathcal{B}_3, \tag{1}$$

where $\mathcal{B}_3$ denotes the set of all analytic functions w in the unit disc such that $w(0)=w'(0)=w''(0)=0$ and $|w(z)|<1$ for $z\in\Delta$. From this, we easily have the following representation for $z/f(z)$:

$$\frac{z}{f(z)}-1=-a_2z-\lambda\int_0^1\frac{w(tz)}{t^2}\,dt. \tag{2}$$

Since $w\in\mathcal{B}_3$, from Schwarz' Lemma it follows that $|w(z)|\leq|z|^3$. From (2), we find that

$$\left|\frac{z}{f(z)}-1\right|\leq|z|\left(|a_2|+\frac{\lambda}{2}|z|^2\right),\quad z\in\Delta. \tag{3}$$

In particular, if a_2 and λ are related by the inequality $|a_2|\leq 1-\lambda/2$, then (3) is equivalent to

$$\left|\frac{f(z)}{z}-\frac{1}{1-|z|^2\left(|a_2|+\frac{\lambda}{2}|z|^2\right)^2}\right|\leq\frac{|z|\left(|a_2|+\frac{\lambda}{2}|z|^2\right)}{1-|z|^2\left(|a_2|+\frac{\lambda}{2}|z|^2\right)^2}. \tag{4}$$

In particular, if $f\in\mathcal{U}_3(\lambda)$, then we have

$$\mathrm{Re}\left(\frac{f(z)}{z}\right)\geq\frac{1}{1+|z|\left(|a_2|+\frac{\lambda}{2}|z|^2\right)}>\frac{1}{1+|a_2|+\frac{\lambda}{2}},\quad z\in\Delta.$$

2. Main theorems

Now we state our main results and their corollaries. The proofs of these theorems will be given in Section 3.

Theorem 1. *Let $f\in\mathcal{U}_3(\lambda)$, $\gamma\in(0,1]$, and*

$$\lambda_*(\gamma,|a_2|)=\frac{-2(1+2\cos(\gamma\pi/2))|a_2|+2\sin(\gamma\pi/2)\sqrt{5+4\cos(\gamma\pi/2)-4|a_2|^2}}{5+4\cos(\gamma\pi/2)}.$$

Then $f\in\mathcal{SS}(\gamma)$ for $0<\lambda\leq\lambda_*(\gamma,|a_2|)$.

For $\gamma=1$, Theorem 1 implies the following

Corollary 1. *If $f\in\mathcal{U}_3(\lambda)$, then $f\in\mathcal{S}^*$ whenever $0<\lambda\leq\frac{-2|a_2|+2\sqrt{5-4|a_2|^2}}{5}$.*

For $a_3=a_2^2=0$, Theorem 1 gives the following

Corollary 2. *If $f(z)=z+\sum_{n=4}^{\infty}a_nz^n$ belongs to $\mathcal{U}(\lambda)$ and $\gamma\in(0,1]$, then $f\in\mathcal{SS}(\gamma)$ whenever $0<\lambda\leq\frac{2\sin(\gamma\pi/2)}{\sqrt{5+4\cos(\gamma\pi/2)}}$.*

Our next result gives condition for functions in $\mathcal{U}_3(\lambda)$ to be starlike of order $\delta(\lambda)$.

Theorem 2. *If $f\in\mathcal{U}_3(\lambda)$ and $a=|f''(0)|/2\leq 1$, then $f\in\mathcal{S}^*(\delta)$ whenever $0<\lambda\leq\lambda(\delta,,a)$, where*

$$\lambda(\delta,a)=\begin{cases}\dfrac{2\sqrt{(1-2\delta)(5-4a^2-2\delta)}-2a(1-2\delta)}{5-2\delta} & \text{if } 0\leq\delta<\dfrac{1+2a}{4+2a},\\[2ex] \dfrac{2-2\delta(1-a)}{2+\delta} & \text{if } \dfrac{1+2a}{4+2a}\leq\delta<\dfrac{1}{1+a}.\end{cases}$$

From Theorem 2, one can obtain a number of new results. For example, if $f\in\mathcal{U}_3(\lambda)$ and $a=|f''(0)/2|\leq 1$ then $f\in\mathcal{S}^*(1/2)$ whenever $0<\lambda\leq 2(1+a)/5$. Moreover, if we choose $a=0$ in Theorem 2, we have