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Noether's symmetry theorem for nabla problems of the calculus of variations

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ABSTRACT

We prove a Noether-type symmetry theorem and a DuBois–Reymond necessary optimality condition for nabla problems of the calculus of variations on time scales.

1. Introduction

The theory of time scales was born with the 1989 Ph.D. Thesis of Stefan Hilger, done under supervision of Bernd Aulbach [1]. The aim was to unify various concepts from the theories of discrete and continuous dynamical systems, and to extend such theories to more general classes of dynamical systems. The calculus of time scales is nowadays a powerful tool, with two excellent books dedicated to it [2,3]. For a good introductory survey on time scales we refer the reader to [4].

The calculus of variations is well-studied in the continuous, discrete, and quantum settings (see, e.g., [5–7]). Recently an important and very active line of research has been unifying and generalizing the known calculus of variations on $\mathbb{R}$, $\mathbb{Z}$, and $q^{\mathbb{N}_0} := \{q^k | k \in \mathbb{N}_0\}$, $q > 1$, to an arbitrary time scale $\mathbb{T}$ via delta calculus. Progress toward this has been made on the topics of necessary and sufficient optimality conditions and its applications—see [8–12] and references therein. The goal is not to simply re-prove existing and well-known theories, but rather to view $\mathbb{R}$, $\mathbb{Z}$, and $q^{\mathbb{N}_0}$ as special cases of a single and more general theory. Doing so reveals richer mathematical structures (cf. [9]), which has great potential for new applications, in particular in engineering [13] and economics [14,15].

The theory of time scales is, however, not unique. Essentially, two approaches are followed in the literature: one dealing with the delta calculus (the forward approach) [2]; the other dealing with the nabla calculus (the backward approach) [3, Chap. 3]. To actually solve problems of the calculus of variations and optimal control it is often more convenient to work backwards in time, and recently a general theory of the calculus of variations on time scales was introduced via the nabla operator. Results include: Euler–Lagrange necessary optimality conditions [16], necessary conditions for higher-order nabla problems [17], and optimality conditions for variational problems subject to isoperimetric constraints [18]. In this note we develop further the theory by proving two of the most beautiful results of the calculus of variations—the Noether symmetry theorem and the DuBois–Reymond condition [19]—for nabla variational problems on an arbitrary time scale $\mathbb{T}$. Our main tool is the recently developed duality technique of Caputo [20], which allows us to obtain nabla results on time scales from the

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delta theory. Caputo's duality concept is briefly presented in Section 2; in Section 3 our results are formulated and proved; in Section 4 an illustrative example is given. We end with some words about the originality of our results and the state of the art (Section 5).

2. Preliminaries

We assume the reader to be familiar with the calculus on time scales [2,3]. Here we just review the main tool used in the work: duality.

Let $\mathbb{T}$ be an arbitrary time scale and let $\mathbb{T}^* := \{s \in \mathbb{R} : -s \in \mathbb{T}\}$. The new time scale $\mathbb{T}^*$ is called the *dual time scale* of $\mathbb{T}$. If σ and ρ denote, respectively, the forward and backward jump operators on $\mathbb{T}$, then we denote by $\widehat{\sigma}$ and $\widehat{\rho}$ the forward and backward jump operators of $\mathbb{T}^*$. Similarly, if μ and ν denote, respectively, the forward and backward graininess function on $\mathbb{T}$, then $\widehat{\mu}$ and $\widehat{\nu}$ denote, respectively, the forward and backward graininess function on $\mathbb{T}^*$; if Δ (resp. ∇) denotes the delta (resp. nabla) derivative on $\mathbb{T}$, then $\widehat{\Delta}$ (resp. $\widehat{\nabla}$) will denote the delta (resp. nabla) derivative on $\mathbb{T}^*$.

Definition 2.1. Given a function $f : \mathbb{T} \to \mathbb{R}$ defined on time scale $\mathbb{T}$ we define the dual function $f^* : \mathbb{T}^* \to \mathbb{R}$ by $f^*(s) := f(-s)$ for all $s \in \mathbb{T}^*$.

We recall some basic results concerning the relationship between *dual objects*. The set of all rd continuous (resp. ld continuous) functions is denoted by C_{rd} (resp. C_{ld}). Similarly, C^1_{rd} (resp. C^1_{ld}) will denote the set of functions from C_{rd} (resp. C_{ld}) whose delta (resp. nabla) derivative belongs to C_{rd} (resp. C_{ld}).

Proposition 2.2 (*[20]*)**.** *Let $\mathbb{T}$ be a given time scale with $a, b \in \mathbb{T}$, $a < b$, and $f : \mathbb{T} \to \mathbb{R}$. Then,*

1. $(\mathbb{T}^\kappa)^* = (\mathbb{T}^*)_\kappa$ *and* $(T_\kappa)^* = (\mathbb{T}^*)^\kappa$;
2. $([a, b])^* = [-b, -a]$ *and* $([a, b]^\kappa)^* = [-b, -a]_\kappa \subseteq \mathbb{T}^*$;
3. *for all* $s \in \mathbb{T}^*$, $\widehat{\sigma}(s) = -\rho(-s) = -\rho^*(s)$ *and* $\widehat{\rho}(s) = -\sigma(-s) = -\sigma^*(s)$;
4. *for all* $s \in \mathbb{T}^*$, $\widehat{\nu}(s) = \mu^*(s)$ *and* $\widehat{\mu}(s) = \nu^*(s)$;
5. f *is* rd *(resp.* ld*) continuous if and only if its dual* $f^* : \mathbb{T}^* \to \mathbb{R}$ *is* ld *(resp.* rd*) continuous;*
6. *if* f *is* delta *(resp.* nabla*) differentiable at* $t_0 \in \mathbb{T}^\kappa$ *(resp. at* $t_0 \in \mathbb{T}_\kappa$*), then* $f^* : \mathbb{T}^* \to \mathbb{R}$ *is* nabla *(resp.* delta*) differentiable at* $-t_0 \in (\mathbb{T}^*)_\kappa$ *(resp.* $-t_0 \in (\mathbb{T}^*)^\kappa$*), and*
$$f^\Delta(t_0) = -(f^*)^{\widehat{\nabla}}(-t_0) \quad (\textit{resp. } f^\nabla(t_0) = -(f^*)^{\widehat{\Delta}}(-t_0)),$$
$$f^\Delta(t_0) = -((f^*)^{\widehat{\nabla}})^*(t_0) \quad (\textit{resp. } f^\nabla(t_0) = -((f^*)^{\widehat{\Delta}})^*(t_0)),$$
$$(f^\Delta)^*(-t_0) = -((f^*)^{\widehat{\nabla}})(-t_0) \quad (\textit{resp. } (f^\nabla)^*(-t_0) = -(f^*)^{\widehat{\Delta}}(-t_0));$$
7. f *belongs to* C^1_{rd} *(resp.* C^1_{ld}*) if and only if its dual* $f^* : \mathbb{T}^* \to \mathbb{R}$ *belongs to* C^1_{ld} *(resp.* C^1_{rd}*);*
8. *if* $f : [a, b] \to \mathbb{R}$ *is* rd *continuous, then*
$$\int_a^b f(t)\Delta t = \int_{-b}^{-a} f^*(s)\widehat{\nabla}s;$$
9. *if* $f : [a, b] \to \mathbb{R}$ *is* ld *continuous, then*
$$\int_a^b f(t)\nabla t = \int_{-b}^{-a} f^*(s)\widehat{\Delta}s.$$

Definition 2.3. Given a Lagrangian $L : \mathbb{T}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$, we define the corresponding dual Lagrangian $L^* : \mathbb{T}^*\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ by $L^*(s, x, v) = L(-s, x, -v)$ for all $(s, x, v) \in \mathbb{T}^* \times \mathbb{R}^n \times \mathbb{R}^n$.

As a consequence of Definition 2.3 and Proposition 2.2 we have the following useful lemma:

Lemma 2.4. *Given a continuous Lagrangian* $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *one has*
$$\int_a^b L\left(t, y^\sigma(t), y^\Delta(t)\right)\Delta t = \int_{-b}^{-a} L^*\left(s, (y^*)^{\widehat{\rho}}(s), (y^*)^{\widehat{\nabla}}(s)\right)\widehat{\nabla}s$$
for all functions $y \in C^1_{\mathrm{rd}}([a, b], \mathbb{R}^n)$.

Definition 2.5. Let $\mathbb{T}$ be a given time scale with at least three points, $n \in \mathbb{N}$, and $L : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ be of class C^1. Suppose that $a, b \in \mathbb{T}$ and $a < b$. We say that $q_0 \in C^1_{\mathrm{rd}}$ is a local minimizer for the problem

$$\mathcal{I}[q] = \int_a^b L(t, q^\sigma(t), q^\Delta(t))\Delta t \longrightarrow \min \tag{1}$$
$$q(a) = q_a, \qquad q(b) = q_b,$$