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Tauberian conditions with controlled oscillatory behavior

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ABSTRACT

Let (u_n) be a sequence of real numbers. In this paper, we give new Tauberian conditions imposed on the general control modulo of the oscillatory behavior of integer order $m \geq 1$ of a sequence (u_n), under which, convergence follows from the Abel summability. These are generalizations of some of the well-known classical Tauberian conditions.

1. Introduction

Throughout this paper, the symbols $u_n = o(1)$, $u_n = O(1)$ mean respectively that $u_n \to 0$ as $n \to \infty$ and that u_n is bounded for n large enough. The classical control modulo of the oscillatory behavior of $u = (u_n)$ is denoted by $\omega_n^{(0)}(u) = n\Delta u_n$, where $\Delta u_n = u_n - u_{n-1}$ and $u_{-1} = 0$. The general control modulo of the oscillatory behavior of integer order $m \geq 1$ of a sequence (u_n) is defined inductively in [1,2] by $\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega^{(m-1)}(u))$, where $\sigma_n^{(1)}(u) = \frac{1}{n+1}\sum_{k=0}^{n} u_k$.

A sequence (u_n) is said to be Abel summable to s, if $\sum_{n=0}^{\infty} \Delta u_n x^n$ converges for $0 < x < 1$, and tends to s as $x \to 1^-$. Every convergent sequence is Abel summable, but the converse is not necessarily true. The Abel summability of (u_n) may imply convergence of (u_n) by adding some suitable conditions on the sequence (u_n). Such a condition is called a Tauberian condition for the Abel summability method and the resulting theorem is called a Tauberian theorem for the Abel summability method.

The Kronecker identity

$$u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u), \tag{1.1}$$

where $V_n^{(0)}(\Delta u) = \frac{1}{n+1}\sum_{k=0}^{n} k\Delta u_k$, is well-known and will be used in the various steps of proofs. Since the arithmetic means of (u_n) can be also expressed as

$$\sigma_n^{(1)}(u) = u_0 + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k},$$

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we may rewrite (1.1) as

$$u_n = V_n^{(0)}(\Delta u) + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k} + u_0.$$

For any sequence (u_n) we define $(n\Delta)_m u_n = (n\Delta)_{m-1}((n\Delta)u_n) = n\Delta((n\Delta)_{m-1}u_n)$ for any positive integer m and non-negative integer n, where $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n$. It is shown in [3] that for each integer $m \geq 1$, $\omega_n^{(m)}(u) = (n\Delta)_m V_n^{(m-1)}(\Delta u)$, where $V_n^{(m-1)}(\Delta u) = \sigma_n^{(1)}(V^{(m-2)}(\Delta u))$. Define $\sigma^{(k)}(u) = (\sigma_n^{(k)}(u))$ by $\sigma_n^{(k)}(u) = \frac{1}{n+1}\sum_{j=0}^{n} \sigma_j^{(k-1)}(u)$ for each integer $k \geq 1$ and $\sigma_n^{(0)}(u) = u_n$.

A sequence (u_n) is called slowly oscillating [4], if

$$u_n - u_m = o(1) \quad \text{as } n > m \to \infty \text{ and } \frac{n}{m} \to 1.$$

For a more functional definition of slow oscillation, see Stanojević [5] and Szász [6].

A sequence (u_n) is said to be slowly oscillating [5] if

$$\lim_{\lambda \to 1^+} \limsup_{n} \max_{n \leq k \leq [\lambda n]} \left| \sum_{j=n+1}^{k} \Delta u_j \right| = 0.$$

A sequence (u_n) is said to be $(C, 1)$ slowly oscillating if $(\sigma_n^{(1)}(u))$ is slowly oscillating.

We now give the one-sided slow oscillation of a sequence.

A sequence (u_n) is said to be one-sidedly slowly oscillating [6], if

$$\lim_{\lambda \to 1^+} \limsup_{n} \sum_{j=n+1}^{[\lambda n]} (|\Delta u_j| - \Delta u_j) = 0.$$

A sequence (u_n) is said to be one-sidedly $(C, 1)$ slowly oscillating if $(\sigma_n^{(1)}(u))$ is one-sidedly slowly oscillating.

Note that every increasing sequence is one-sidedly slowly oscillating and each one-sidedly slowly oscillating sequence is not necessarily increasing.

A sequence (u_n) is said to be slowly decreasing [7], if for $\lambda > 1$,

$$\lim_{\lambda \to 1^+} \liminf_{n} \min_{n \leq k \leq [\lambda n]} \sum_{j=n+1}^{k} \Delta u_j \geq 0.$$

A sequence (u_n) is said to be $(C, 1)$ slowly decreasing if $(\sigma_n^{(1)}(u))$ is slowly decreasing.

The following basic conclusions are obtained from the above definitions.

If $\omega_n^{(0)}(u) = O(1)$, then (u_n) is slowly oscillating, and if $\omega_n^{(0)}(u) \geq 0$, then (u_n) is slowly decreasing. Dik [1] proved that a sequence (u_n) is slowly oscillating if and only if $(V_n^{(0)}(\Delta u))$ is slowly oscillating and $V_n^{(0)}(\Delta u) = O(1)$. From this, one concludes that $(\sigma_n^{(1)}(u))$ is slowly oscillating for every slowly oscillating (u_n). Szász [6] showed that if a (u_n) is one-sidedly slowly oscillating, then $V_n^{(0)}(\Delta u) \geq -C$. for some non-negative C. From Szász's result it easily follows that if a sequence (u_n) is one-sidedly slowly oscillating, then $(\sigma_n^{(1)}(u))$ is slowly decreasing.

Schmidt [4] proved that slow oscillation of (u_n) is a Tauberian condition for the Abel summability method. Szász [8] and Jakimovski [9] improved Schmidt's Tauberian condition that $(\omega_n^{(0)}(u))$ is one-sidedly $(C, 1)$ slowly oscillating and $(\omega_n^{(0)}(u))$ is $(C, 1)$ slowly decreasing, respectively. Dik [1] proved that $(C, 1)$ slow oscillation of $(\omega_n^{(1)}(u))$ is a Tauberian condition for the Abel summability method. Çanak and Totur [10] replaced Dik's condition by $(C, 1)$ slow oscillation of $(\omega_n^{(m)}(u))$ for any non-negative integer m. Furthermore, Çanak and Totur [3], Çanak [11], Çanak and Totur [12], Çanak et al. [13] have obtained new Tauberian conditions in terms of the general control modulo of the oscillatory behavior of integer order $m \geq 1$ of a sequence (u_n) under which convergence follows from the Cesàro and Abel summability of (u_n).

The main goal of this work is to obtain new Tauberian conditions improving the conditions given by Çanak and Totur [10]. These conditions on the general control modulo of the oscillatory behavior of a sequence also generalize those given by Schmidt [4], Szász [8], and Jakimovski [9].

The following two o-theorems for the Abel summability method are due to Tauber [14].

Theorem 1.1. *Let (u_n) be Abel summable to s. If $\omega_n^{(0)}(u) = o(1)$, then (u_n) converges to s.*

Theorem 1.2. *Let (u_n) be Abel summable to s. If $\sigma_n^{(1)}(\omega^{(0)}(u)) = o(1)$, then (u_n) converges to s.*

A generalization of Theorem 1.1 is given by Hardy–Littlewood [15].

Theorem 1.3. *Let (u_n) be Abel summable to s. If $\omega_n^{(0)}(u) \geq -C$ for some $C > 0$, then (u_n) converges to s.*