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Characteristic decomposition of the 2×2 quasilinear strictly hyperbolic systems ☆

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ABSTRACT

This paper is devoted to extending the well-known result on reducible equations in Courant and Friedrichs' book "Supersonic flow and shock waves", that any hyperbolic state adjacent to a constant state must be a simple wave. We establish a nice sufficient condition for the existence of characteristic decompositions to the general 2×2 quasilinear strictly hyperbolic systems. These decompositions allow for a proof that any wave adjacent to a constant state is a simple wave, despite the fact that the coefficients depend on the independent variables. Consequently as applications, we obtain the same results for the pseudo-steady Euler equations.

1. Introduction

It is well known that the Riemann invariants are useful in the construction of solutions for hyperbolic systems, for example, the construction of d'Alembert formula for the wave equation, and the proof of development of singularities [1]. For the system of two-dimensional steady isentropic irrotational Euler equations, any solution adjacent to a constant state is a simple wave by the existence of Riemann invariants [2]. They also can be used to study the interaction of two rarefaction waves [2]. Another important technique that can be used to study the quasilinear hyperbolic systems is the characteristic decomposition. By using the characteristic decomposition technique, one may find not only Riemann invariants (e.g., homogeneous reducible quasilinear hyperbolic systems), but also the so called Riemann variants in some hyperbolic systems [3,4]. The method of characteristic decomposition is quite effective to deal with some problems for quasilinear hyperbolic systems [5,3,6–11].

The idea to use the characteristic decomposition can trace back to the classical one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0,$$

where the wave speed c is constant. The above equation has an interesting decomposition

$$(\partial_t \pm c\partial_x)(\partial_t \mp c\partial_x)u = 0.$$

Denote

$$R = \partial_- u := \partial_t u - c\partial_x u, \qquad S = \partial_+ u := \partial_t u + c\partial_x u.$$

Then

$$\partial_+ R = 0, \qquad \partial_- S = 0,$$

which imply that R and S are constants along the plus and minus family of characteristics, respectively.

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The characteristic decomposition as a powerful tool for building patches of global smooth solutions was first revealed for the pressure-gradient system by Dai and Zhang [12] and then was used extensively for studying other systems, for example, the Euler equations. It is essential that not only in the elementary wave patches, but also in providing a passage to derive a priori estimates of solutions, see [7,9,11,4,3,8,10,6]. The concept of the characteristic decomposition indeed has more implications, and it can also apply to high order estimates of solutions, even to numerical schemes [13].

The main purpose of the present paper is to extend the well-known theorem of Courant and Friedrichs [2], that any hyperbolic state adjacent to a constant state must be a simple wave, to the results for non-reducible equations. A solution is called a *simple wave* if it depends only on a single parameter. It plays an important role in the theories of gas dynamics and fluid mechanics. "Simple waves play a fundamental role in describing and building up solutions of flow problems". (pp. 59–60, Courant and Friedrichs [2]). The main tool of the analysis in the present paper is the characteristic decomposition. A nice sufficient condition for the existence of characteristic decompositions to the general 2×2 quasilinear hyperbolic systems is established in Section 2. Using these decompositions, we extend the theorem of Courant and Friedrichs, despite the fact that the coefficients depend on the independent variables. The applications to the pseudo-steady Euler equations and the generalized UTSD system are given, and the same results as in [14,5] are obtained in Section 3.

2. Two-by-two system

Consider a 2×2 strictly hyperbolic system

$$\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_y = 0, \tag{2.1}$$

where the coefficients $a_{ij} = a_{ij}(x, y, u, v)$, $i, j = 1, 2$. The coefficient matrix has two eigenvalues

$$\lambda_\pm = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2},$$

which are solutions to the characteristic equation

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0. \tag{2.2}$$

We have the left eigenvectors

$$l_\pm = \left(1, \frac{\lambda_\pm - a_{11}}{a_{21}}\right), \tag{2.3}$$

provided that $a_{21} \neq 0$ at first. The case $a_{21} = 0$ will be considered later. The characteristic form of (2.1) is therefore

$$\partial_\pm u + \frac{\lambda_\mp - a_{22}}{-a_{21}} \partial_\pm v = 0, \tag{2.4}$$

where $\partial_\pm := \partial_x + \lambda_\pm \partial_y$. We then compute

$$\begin{aligned} \partial_\pm \lambda_\pm &= \lambda_{\pm u}\partial_\pm u + \lambda_{\pm v}\partial_\pm v + \lambda_{\pm x}\partial_\pm x + \lambda_{\pm y}\partial_\pm y \\ &= \left(\lambda_{\pm u} + \frac{a_{21}}{\lambda_\mp - a_{22}}\lambda_{\pm v}\right)\partial_\pm u + \lambda_{\pm x} + \lambda_\pm \lambda_{\pm y}. \end{aligned} \tag{2.5}$$

If the following equations:

$$\lambda_{\pm x} + \lambda_\pm \lambda_{\pm y} = 0 \tag{2.6}$$

hold, then we find that

$$\partial_\pm \lambda_\pm = \left(\lambda_{\pm u} + \frac{a_{21}}{\lambda_\mp - a_{22}}\lambda_{\pm v}\right)\partial_\pm u.$$

We now compute the derivative of second order $\partial_\pm \partial_\mp u$. At first, we need the following commutator relation [3]:

Lemma 1 (*Commutator Relation*)**.** *For any quantity $I = I(x, y)$, there holds*

$$\partial_- \partial_+ I - \partial_+ \partial_- I = \frac{\partial_- \lambda_+ - \partial_+ \lambda_-}{\lambda_- - \lambda_+}(\partial_- I - \partial_+ I). \tag{2.7}$$