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Mean-field limit for the stochastic Vicsek model

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ABSTRACT

We consider the continuous version of the Vicsek model with noise, proposed as a model for collective behaviour of individuals with a fixed speed. We rigorously derive the kinetic mean-field partial differential equation satisfied when the number N of particles tends to infinity, quantifying the convergence of the law of one particle to the solution of the PDE. For this we adapt a classical coupling argument to the present case in which both the particle system and the PDE are defined on a surface rather than on the whole space $\mathbb{R}^d$. As part of the study we give existence and uniqueness results for both the particle system and the PDE.

1. Introduction

The stochastic Vicsek model [1] arises in the study of collective motion of animals and it is receiving lots of attention due to the appearance of a phase transition [2,3]. A continuum version and variants of this model have been proposed in the recent works [4,5]. Our objective is to rigorously derive some continuum partial differential equations analysed in [4] from the stochastic Vicsek particle model. This was carried out for a family of collective behaviour models in [6] following the method of [7]. The present models do not fall into this analysis due to the evolution being defined on a surface as we explain next. In the models considered here, individuals are assumed to move with a fixed cruising speed trying to average their orientations with other individuals in the swarm in the presence of noise. This orientation mechanism is modelled by locally averaging in space their relative velocity to other individuals. More precisely, we are interested in the behaviour of N interacting $\mathbb{R}^{2d}$-valued processes $(X_t^i, V_t^i)_{t\geq 0}$ with $1 \leq i \leq N$ with constant speed $|V_t^i|$, say unity. We define them as solutions to the coupled Stratonovich stochastic differential equations

$$\begin{cases} dX_t^i = V_t^i\, dt, \\ dV_t^i = \sqrt{2}\, P(V_t^i) \circ dB_t^i - P(V_t^i) \left(\dfrac{1}{N} \displaystyle\sum_{j=1}^N K(X_t^i - X_t^j)(V_t^i - V_t^j) \right) dt. \end{cases} \tag{1}$$

Here $P(v)$ is the projection operator on the tangent space at $v/|v|$ to the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, i.e.,

$$P(v) = I - \frac{v \otimes v}{|v|^2}.$$

This stochastic system is considered with independent and commonly distributed initial data $(X_0^i, V_0^i) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$ with $1 \leq i \leq N$. The $(B_t^i)_{t\geq 0}$ denote N independent standard Brownian motions in $\mathbb{R}^d$. The projection operator ensures that V_t^i

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keeps constant norm, equal to 1. The second term in the evolution of V_t^i models the tendency of the particle i to have the same orientation as the other particles, in a way weighted by the interaction kernel K, as in the model proposed by Cucker and Smale [8]. Let us observe that $P(V_t^i)V_t^i = 0$, so we can drop the corresponding term when writing (1) to recover the usual formulations as in [4].

We will work with stochastic processes defined on $\mathbb{R}^{2d}$ instead of $\mathbb{R}^d \times \mathbb{S}^{d-1}$. We will check later on that solutions of (1) with initial data in $\mathbb{R}^d \times \mathbb{S}^{d-1}$ remain there for all times. We have written (1) in the Stratonovich sense, since the term involving noise corresponds to Brownian motion on the sphere $\mathbb{S}^{d-1}$ as in [9, Section 1.4] and [10, Section V.31].

By symmetry of the initial configuration and of the evolution, all particles have the same distribution. Even though they are initially independent, correlation builds up in time due to the interaction term. Nevertheless, this interaction term is of order $1/N$, and thus, it seems reasonable that two of these interacting particles (or a fixed number k of them) become less and less correlated as N gets large (propagation of chaos).

Following [7] we shall show that the N interacting processes $(X_t^i, V_t^i)_{t\geq 0}$ respectively behave as $N \to \infty$ like the auxiliary processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$, solutions to

$$\begin{cases} d\overline{X}_t^i = \overline{V}_t^i\, dt, \\ d\overline{V}_t^i = \sqrt{2}\, P(\overline{V}_t^i) \circ dB_t^i - P(\overline{V}_t^i)(H * f_t)(\overline{X}_t^i, \overline{V}_t^i)\, dt, \\ (\overline{X}_0^i, \overline{V}_0^i) = (X_0^i, V_0^i), \qquad f_t = \mathrm{law}(\overline{X}_t^i, \overline{V}_t^i) \end{cases} \tag{2}$$

in the Stratonovich sense. Here the Brownian motions $(B_t^i)_{t\geq 0}$ are those governing the evolution of the $(X_t^i, V_t^i)_{t\geq 0}$ and

$$(H * f)(x, v) = \int_{\mathbb{R}^{2d}} K(x - x')\,(v - v')\,f(x', v')\,dx'\,dv', \quad x, v \in \mathbb{R}^d.$$

Note that (2) consists of N equations which can be solved independently of each other. Each of them involves the condition that f_t is the distribution of $(\overline{X}_t^i, \overline{V}_t^i)$, thus making it nonlinear. The processes $(\overline{X}_t^i, \overline{V}_t^i)_{t\geq 0}$ with $i \geq 1$ are independent since the initial conditions and driving Brownian motions are independent.

We will show that these processes defined on $\mathbb{R}^{2d}$ are identically distributed, take values in $\mathbb{R}^d \times \mathbb{S}^{d-1}$ if initially so, and their common law f_t at time t, as a measure on $\mathbb{R}^d \times \mathbb{S}^{d-1}$, evolves according to

$$\partial_t f_t + \omega \cdot \nabla_x f_t = \Delta_\omega f_t + \nabla_\omega \cdot (f_t(I - \omega \otimes \omega)(H * f_t)), \quad t > 0,\ x \in \mathbb{R}^d,\ \omega \in \mathbb{S}^{d-1}. \tag{3}$$

Now the convolution $H * f$ is over $\mathbb{R}^d \times \mathbb{S}^{d-1}$:

$$(H * f)(x, \omega) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} K(x - x')\,(\omega - \omega')\,f(x', \omega')\,dx'\,d\omega', \quad x \in \mathbb{R}^d,\ \omega \in \mathbb{S}^{d-1}.$$

Moreover, ∇_x stands for the gradient with respect to the position variable $x \in \mathbb{R}^d$ whereas ∇_ω, $\nabla_\omega \cdot$ and Δ_ω respectively stand for the gradient, divergence and Laplace–Beltrami operators with respect to the velocity variable $\omega \in \mathbb{S}^{d-1}$.

This equation is proposed in [5] as a continuous version of the original Vicsek model [1], and one of our purposes is to make this derivation rigorous. The asymptotic behaviour and the appearance of a phase transition in the space-homogeneous version of (3) (i.e., without the space variable) has been recently studied in [11]. It is also known as the Doi–Onsager equation, introduced by Doi in [12] as a model for the non-equilibrium Statistical Mechanics of a suspension of polymers in which their spatial orientation (given by the parameter $\omega \in \mathbb{S}^{d-1}$) is taken into account.

The main result of this paper can be summarized as follows.

Theorem 1. *Let f_0 be a probability measure on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ with finite second moment in $x \in \mathbb{R}^d$ and let (X_0^i, V_0^i) for $1 \leq i \leq N$ be N independent variables with law f_0. Let also K be a Lipschitz and bounded map on $\mathbb{R}^d$. Then,*

- (i) *There exists a pathwise unique global solution to the SDE system (1) with initial data (X_0^i, V_0^i) for $1 \leq i \leq N$; moreover, the solution is such that all V_t^i have norm 1.*
- (ii) *There exists a pathwise unique global solution to the nonlinear SDE (2) with initial datum (X_0^i, V_0^i); moreover, the solution is such that $\overline{V}_t^i$ has norm 1.*
- (iii) *There exists a unique global weak solution to the nonlinear PDE (3) with initial datum f_0. Moreover, it is the law of the solution to (2).*

Solutions to general SDE's can be built in submanifolds of $\mathbb{R}^d$ by means of the Brownian motion of the ambient space as in [10, Theorem V.34.86] for instance; then one can interpret the generator in terms of the corresponding Laplace–Beltrami operator. For example, the Brownian motion on a submanifold Σ of $\mathbb{R}^d$ is the solution to the SDE

$$dW_t = P_\Sigma(W_t) \circ dB_t$$

on $\mathbb{R}^d$ and with $P_\Sigma(w)$ being the orthogonal projection of $\mathbb{R}^d$ onto the tangent space at w to Σ. Here, we give the full construction and derivation of the evolution of the law as it can be done explicitly in the case of the sphere $\mathbb{S}^{d-1}$. Let us also emphasize that we have only partial diffusion since it is a kinetic model.