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# Polynomial integration over the unit circle

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ABSTRACT

In this work, recursive formulas facilitating the computation of very complex tensor integrals over the unit circle, are obtained by making use of the divergence theorem and mathematical induction. Using these formulas, the integrals over the unit circle of polynomials with any high degree are easily computed.



## 1. Introduction

The mathematical formulation of many two-dimensional problems in applied mathematics and physics often leads to linear ordinary or partial differential equations and other boundary value problems [1]. The solutions to many of these problems require the computation of a contour integral over the unit circle. For example, one can cite the solution of two-dimensional Laplace equations, expressed in terms of Green's function [1], for problems encountered in electrostatics, elastostatics and heat flow among other physical problems. Li et al. [2] and Wang et al. [3] studied the problem of a two dimensional circular inclusion embedded within a finite circular representative volume element. They computed the analytical expressions of the so-called Dirichlet–Eshelby tensor [2] and the Neumann–Eshelby tensor [3], respectively. Their computations are based on the evaluation of polynomial integrals over the unit circle.

The polynomial integration over the unit circle is feasible for the case of polynomials with low degree, but when the degree increases, the computation becomes intricate. In a previous paper by the first author [4], recursive formulas, facilitating the integration of high order polynomials over the unit sphere have been obtained. These formulas are only applicable under three-dimensional configuration. In the present work, we will follow the same technique as used in [4] to provide formulas allowing the integration of polynomials with any high degree over the unit circle for two-dimensional configuration.

## 2. Preliminaries

Consider a two-dimensional configuration. A point $P$ is defined in the global frame of reference $R = (O, \underline{e}_1, \underline{e}_2)$, by

$$\underline{OP} = x_i \underline{e}_i, \tag{1}$$

where $(x_1, x_2)$ are the global coordinates. Einstein's convention (see [5]) is used.

By making use of the polar coordinates $(r, \theta)$, the position of the point $P$ is expressed by

$$\underline{OP} = r\left[\cos\theta\,\underline{e}_1 + \sin\theta\,\underline{e}_2\right]. \tag{2}$$

Consider a unit circle $C^1$, with surface $S^1$, centered at the origin $O$. The unit vector $\underline{X}$ normal to $C^1$ is defined by

$$\underline{X} = x_i \underline{e}_i = \cos\theta\,\underline{e}_1 + \sin\theta\,\underline{e}_2. \tag{3}$$

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A polynomial Q of order $M$ is the sum of $M+1$ terms with each term defined by

$$\underbrace{A_{ijmn\cdots s}}_{k \text{ indices}} \underbrace{x_i x_j x_m x_n \cdots x_s}_{k \text{ products}} \quad 0 \le k \le M. \tag{4}$$

$A_{ijmn\cdots s}$ is a tensor of order $k$, defining the constant coefficients of $Q$. It contracts with the tensor product of $k$ vectors $x_i, x_j, x_m, x_n, \ldots, x_s$. The contour integral over $C^1$, of (4), can be seen as the tensor's contraction between $A_{ijmn\cdots s}$ and the tensor $I^{C^1,(k)}_{ijmn\cdots s}$ of order $(k)$, defined by the following expression

$$\underbrace{I^{C^1,(k)}_{ijmn\cdots s}}_{k \text{ indices}} = \oint_{C^1} \underbrace{x_i x_j x_m x_n \cdots x_s}_{k \text{ products}} \, dl = \oint_{C^1} \underbrace{X_i X_j X_m X_n \cdots X_s}_{k \text{ products}} \, dl. \tag{5}$$

Consequently, the computation of (5) is needed for the integration of (4) over $C^1$.

## 3. Main results

**Proposition 1.** *The contour integral over $C^1$ and surface integral over $S^1$ of the tensor product of k vectors $x_i, x_j, x_m, x_n, \ldots, x_s$, respectively $I^{C^1,(k)}_{ijmn\cdots s}$ and $I^{S^1,(k)}_{ijmn\cdots s}$ are related to the following relation:*

$$I^{S^1,(k)}_{ijmn\cdots s} = \frac{1}{(k+2)} I^{C^1,(k)}_{ijmn\cdots s} \tag{6}$$

*with,* $\underbrace{I^{S^1,(k)}_{ijmn\cdots s}}_{k \text{ indices}} = \int_{S^1} \underbrace{x_i x_j x_m x_n \cdots x_s}_{k \text{ products}} \, ds$ *and k is an even integer.*

In the case where $k$ is an odd integer, $I^{S^1,(k)}_{ijmn\cdots s}$ and $I^{C^1,(k)}_{ijmn\cdots s}$ are zero.

**Proof.** The demonstration is based on mathematical induction:

If $k=0$, we have the following

$$I^{S^1,(0)} = \int_{S^1} ds = \pi = \frac{1}{(2+0)}(2\pi) = \frac{1}{(2+0)} \oint_{C^1} dl = \frac{1}{(2+0)} I^{C^1,(0)}. \tag{7}$$

If $k=1$

$$I^{S^1,(1)}_i = \int_{S^1} x_i \, ds = \oint_{C^1} x_i \, dl = I^{C^1,(1)}_i = 0. \tag{8}$$

Suppose, if $k$ is an even integer,

$$I^{S^1,(k)}_{ijmn\cdots s} = \frac{1}{(k+2)} I^{C^1,(k)}_{ijmn\cdots s} \tag{9}$$

and that $I^{C^1,(k+1)}_{ijmn\cdots s}, I^{S^1,(k+1)}_{ijmn\cdots s}$ are zero.

Let us prove the following two points

- $$I^{S^1,(k+2)}_{ijmn\cdots s} = \frac{1}{(k+4)} I^{C^1,(k+2)}_{ijmn\cdots s}. \tag{10}$$
- $I^{C^1,(k+3)}_{ijmn\cdots s}, I^{S^1,(k+3)}_{ijmn\cdots s}$ are zero.

To prove the first point, the tensor $I^{S^1,(k+2)}_{ijmn\cdots s}$ is written as follows

$$\begin{aligned} I^{S^1,(k+2)}_{ijmn\cdots s} &= \int_{S^1} \underbrace{x_i x_j \cdots x_s}_{k+2} \, ds \\ &= \int_0^1 r \, dr \int_0^{2\pi} \underbrace{x_i x_j \cdots x_s}_{k+2} \, d\theta \\ &= \int_0^1 r \, dr \int_0^{2\pi} r^{k+2} \underbrace{X_i X_j \cdots X_s}_{k+2} \, d\theta \end{aligned}$$