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# Sequential definitions of connectedness

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ABSTRACT

We investigate the impact of changing the definition of convergence of sequences on the structure of the set of connected subsets of a topological group, $X$. A non-empty subset $A$ of $X$ is called $G$-sequentially connected if there are no non-empty and disjoint $G$-sequentially closed subsets $U$ and $V$, both meeting A, such that $A \subseteq U \bigcup V$. Sequential connectedness in a topological group is a special case of this generalization when $G = \lim$.



## 1. Introduction

Connectedness and other related concepts play a very important role not only in pure mathematics but also in other branches of science involving mathematics, especially in geographic information systems, population modelling, and motion planning in robotics.

Connor and Grosse-Erdmann [1] have investigated the impact of changing the definition of the convergence of sequences on the structure of sequential continuity of real functions. Çakallı [2] extended this concept to the topological group setting, introducing the concept of $G$-sequential compactness. He reported further results on $G$-sequential compactness and $G$-sequential continuity in [3]. One is often relieved to find that the standard closed-set definition of connectedness for metric spaces can be replaced by a sequential definition of connectedness. That many of the properties of connectedness of sets can be easily derived using sequential arguments has also been, no doubt, a source of relief to the interested mathematics instructor.

The aim of this paper is to introduce $G$-sequential connectedness and to investigate the concept in metrisable topological groups.

## 2. Preliminaries

Before we begin, we will state some definitions and notation. Throughout this paper, $\mathbf{N}$ will denote the set of all positive integers. Although some of the definitions that follow make sense for an arbitrary topological group, we prefer using neighbourhoods instead of metrics. In this paper, $X$ will always denote a topological Hausdorff group, written additively, that satisfies the first axiom of countability. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of terms of $X$. The notations $s(X)$ and $c(X)$ denote the set of all $X$-valued sequences and the set of all $X$-valued convergent sequences of points in $X$, respectively.

Following the idea given in a 1946 American Mathematical Monthly problem [4], a number of authors, Posner [5], Iwinski [6], Srinivasan [7], Antoni [8], Antoni and Salat [9], and Spigel and Krupnik [10], have studied $A$-continuity defined

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by a regular summability matrix $A$. Some authors, Öztürk [11], Savaş [12], Savaş and Das [13], and Borsik and Salat [14] have studied $A$-continuity for methods of almost convergence or for related methods. See also [15] for an introduction to summability matrices.

The idea of statistical convergence was formerly described under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [16]. The concept was formally introduced by Fast [17] and was later reintroduced by Schoenberg [18] and also, independently, by Buck [19]. A sequence $(x_k)$ of points in $X$ is said to be statistically convergent to an element $\ell$ of $X$ if for each neighbourhood $U$ of 0,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : x_k - \ell \notin U\}| = 0,$$

and this is denoted by $st-\lim_{n\to\infty} x_n = \ell$. The statistical limit is an additive function on the group of statistically convergent sequences of points in $X$. (See [20] for the real case and [21–25] for the topological group setting.)

A sequence $(x_k)$ of points in a topological group is called lacunary statistically convergent to an element $\ell$ of $X$ if

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : x_k - \ell \notin U\}| = 0$$

for every neighbourhood $U$ of 0 where $I_r = (k_{r-1}, k_r]$ and $k_0 = 0$, $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$, and $\theta = (k_r)$ is an increasing sequence of positive integers. For a constant lacunary sequence, $\theta = (k_r)$, the lacunary statistically convergent sequences in a topological group form a subgroup of the group of all $X$-valued sequences, and the lacunary statistical limit is an additive function on this space. (See [21] for the topological group setting, and see [26,27] for the real case.) Throughout this paper, we assume that $\lim\inf_r \frac{k_r}{k_{r-1}} > 1$.

By a method of sequential convergence, or briefly by a method, we mean an additive function $G$ defined on a subgroup $c_G(X)$ of $s(X)$ into $X$ [2]. A sequence $\mathbf{x} = (x_n)$ is said to be $G$-convergent to $\ell$ if $\mathbf{x} \in c_G(X)$ and $G(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the group $c(X)$. A method $G$ is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is $G$-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. Clearly, if $f$ is $G$-sequentially continuous on $X$, then it is $G$-sequentially continuous on every subset $Z$ of $X$, but the converse is not necessarily true, as in the latter case, the sequences $x$ are restricted to $Z$. This distinction was demonstrated by an example in [1] for a real function.

We define the sum of two methods of sequential convergence, $G_1$ and $G_2$, as

$$(G_1 + G_2)(\mathbf{x}) = G_1(\mathbf{x}) + G_2(\mathbf{x}),$$

where $c_{G_1+G_2}(X) = c_{G_1}(X) \cap c_{G_2}(X)$ [3].

The notion of regularity introduced above coincides with the classical notion of regularity for summability matrices and with regularity in a topological group for limitation methods. See [15] for an introduction to regular summability matrices of real numbers and complex numbers, [28] for an introduction to regular limitation (summability) methods in topological groups, and [29] for a general view of sequences of real or complex numbers.

First of all, we recall the definition of $G$-sequential closure of a subset of $X$ [2,3]. Let $A \subset X$ and $\ell \in X$. Then $\ell$ is in the $G$-sequential closure of $A$ (called the $G$-hull of $A$ in [1]) if there is a sequence $\mathbf{x} = (x_n)$ of points in $A$ such that $G(\mathbf{x}) = \ell$. We denote the $G$-sequential closure of a set $A$ by $\overline{A}^G$. We say that a subset $A$ is *$G$-sequentially closed* if it contains all of the points in its $G$-closure, i.e., a subset $A$ of $X$ is $G$-sequentially closed if $\overline{A}^G \subset A$. The null set $\phi$ and the whole space $X$ are $G$-sequentially closed.

It is clear that $\overline{\phi}^G = \phi$ and $\overline{X}^G = X$ for a regular method $G$. If $G$ is a regular method, then $A \subset \overline{A} \subset \overline{A}^G$, and hence $A$ is $G$-sequentially closed if and only if $\overline{A}^G = A$. Even for regular methods, it is not always true that $\overline{(\overline{A}^G)}^G = \overline{A}^G$.

Even for regular methods, the union of any two $G$-sequentially closed subsets of $X$ need not be a $G$-sequentially closed subset of $X$, as is seen by considering Counterexample 1 given after Theorem 4 in [3].

Çakallı introduced the concept of $G$-sequential compactness and proved that a $G$-sequentially continuous image of any $G$-sequentially compact subset of $X$ is also $G$-sequentially compact (see Theorem 7 in [2]). He investigated $G$-sequential continuity and obtained further results in [3] (See also [30–37] for some other types of continuities that cannot be given by any sequential method).

Recently, Mucuk and Sahan [38] investigated further properties of $G$-sequential closed subsets of $X$. They modified the definition of the open subset to the $G$-sequential case in the sense that a subset $A$ of $X$ is *$G$-sequentially open* if its complement is $G$-sequentially closed, i.e., $\overline{X \setminus A}^G \subseteq X \setminus A$, and obtained that the union of any $G$-sequentially open subsets of $X$ are $G$-sequentially open. From the fact that the $G$-sequential closure of a subset of $X$ includes the set itself for a regular sequential method $G$, we see that a subset $A$ is $G$-sequentially open if and only if $\overline{X \setminus A}^G = X \setminus A$ for a regular sequential method $G$. A function $f : X \to X$ is *$G$-sequentially continuous* at a point $u$ if, given a sequence $(x_n)$ of points in $X$, $G(\mathbf{x}) = u$ implies that $G(f(\mathbf{x})) = f(u)$. If a function $f$ is $G$-sequentially continuous on $X$, then the inverse image $f^{-1}(K)$ of any $G$-sequentially closed subset $K$ of $X$ is $G$-sequentially closed [2]. If a function $f$ is $G$-sequentially continuous on $X$, then the inverse image $f^{-1}(U)$ of any $G$-sequentially open subset $U$ of $X$ is $G$-sequentially open [38]. For a regular method $G$, the function $f_a : X \to X$, $x \mapsto a+x$ is $G$-sequentially continuous, $G$-sequentially closed, and $G$-sequentially open [38]. If $A$ and $B$ are $G$-sequentially open, then so is the sum $A + B$ [38]. Mucuk and Sahan also obtained that a subset $A$ of $X$ is $G$-sequentially open if and only if each $a \in A$ has a $G$-sequentially open neighbourhood $U_a$ such that $U_a \subseteq A$.