Applied Mathematics Letters 25 (2012) 505–508

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A note on Jordan type inequalities for hyperbolic functions

Yupei Lv [a,*], Gendi Wang [a,b], Yuming Chu [a]

[a] Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China

[b] College of Mathematics and Econometrics, Hunan University, Changsha 410082, China

ARTICLE INFO

Article history:
Received 4 September 2010
Received in revised form 5 August 2011
Accepted 23 September 2011

Keywords:
Jordan type inequalities
Hyperbolic

ABSTRACT

In this work the authors present some new lower and upper bounds for the functions $\sin x/x$ and $x/\sinh x$, thus improving some inequalities put forward by Klén et al. (2010) in the paper [2].

1. Introduction

The well-known Jordan inequality [1] can be stated as follows:

$$\frac{2}{\pi}x \le \sin x \le x, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{1.1}$$

During the past few years the classical Jordan inequality has been the focus of studies on trigonometric inequalities and many refinements have been proved (cf. [1–8]). For a long list of recent papers on this topic see [9].

Very recently Klén et al. [2] have dealt with the Jordan type inequalities for hyperbolic functions and proved the inequalities

$$\cos^2\frac{x}{2} \le \frac{\sin x}{x} \le \cos^3\frac{x}{3}, \quad x \in (-\sqrt{27/5}, \sqrt{27/5}), \tag{1.2}$$

and

$$\left(\frac{1}{\cosh x}\right)^{1/2} < \frac{x}{\sinh x} < \left(\frac{1}{\cosh x}\right)^{1/4}, \quad x \in (0, 1). \tag{1.3}$$

This work is motivated by these studies and we aim to refine the above inequalities.

2. The main results and proofs

The following l'Hôpital type criterion for monotonicity of the quotient of two functions from [10, Theorem 1.25] plays an essential role in all the above mentioned studies. For some other applications of this criterion see [10–13].

* Corresponding author.
E-mail address: xhzhang_80@126.com (Y. Lv).

0893-9659/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2011.09.046

Lemma 1 (*The Monotone Form of l'Hôpital's Rule [10]*). *For $-\infty < a < b < \infty$, let $f, g: [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and let $g'(x) \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing or decreasing on (a, b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Now we state the main results and their proofs.

Theorem 1. *If $x \in (0, \pi/2)$ and $a = (\log(\pi/2))/\log\sqrt{2} \approx 1.30299$, then*

$$\cos^{\frac{4}{3}}\frac{x}{2} < \frac{\sin x}{x} < \cos^{a}\frac{x}{2} \tag{2.1}$$

with the best possible constants a and $4/3$. □

Proof. Let $f(x) = (\log(\sin x/x))/\log\cos(x/2) \equiv f_1(x)/f_2(x)$, where $f_1(x) = \log(\sin x/x)$ and $f_2(x) = \log\cos(x/2)$, with $f_1(0+) = f_2(0+) = 0$. By differentiation we have

$$\frac{f_1'(x)}{f_2'(x)} = 2\frac{\sin x - x\cos x}{x - x\cos x} \equiv 2\frac{f_3(x)}{f_4(x)},$$

where $f_3(x) = \sin x - x\cos x$ and $f_4(x) = x - x\cos x$, with $f_3(0) = f_4(0) = 0$. Differentiation gives

$$\frac{f_3'(x)}{f_4'(x)} = \frac{1}{1 + (1-\cos x)/(x\sin x)} = \frac{1}{1 + f_5(x)/f_6(x)},$$

where $f_5(x) = 1 - \cos x$ and $f_6(x) = x\sin x$, with $f_5(0) = f_6(0) = 0$. Then we get

$$\frac{f_5'(x)}{f_6'(x)} = \frac{1}{1 + x/\tan x} = \frac{1}{1 + f_7(x)/f_8(x)},$$

where $f_7(x) = x$ and $f_8(x) = \tan x$, with $f_7(0) = f_8(0) = 0$. By differentiation we have $f_7'(x)/f_8'(x) = \cos^2 x$, which is clearly decreasing. By Lemma 1, $f(x)$ is strictly decreasing. Clearly, $f(\pi/2-) = (\log(\pi/2))/\log\sqrt{2} \equiv a \approx 1.30299$. By l'Hôpital rule, $f(0+) = 4/3$. The inequality follows from the monotonicity of f. □

In [2, Theorem 3.7], the bounds of $\cosh x$ were given as follows:

$$\left(\frac{1}{\cos x}\right)^{2/3} < \cosh x < \frac{1}{\cos x}, \quad x \in (0, \pi/4).$$

Now we refine the lower bound of $\cosh x$.

Theorem 2. *If $x \in (0, \pi/4)$ and $a = \log\cosh(\pi/4)/\log\sqrt{2} \approx 0.811133$, then*

$$\left(\frac{1}{\cos x}\right)^{a} < \cosh x < \frac{1}{\cos x} \tag{2.2}$$

with the best possible constants a and 1.

Proof. Let $f(x) = (\log\cosh x)/\log(1/\cos x) \equiv f_1(x)/f_2(x)$, where $f_1(x) = \log\cosh x$ and $f_2(x) = \log(1/\cos x)$, with $f_1(0) = f_2(0) = 0$. By differentiation we have

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\tanh x}{\tan x} \equiv f_3(x).$$

$$f_3'(x) = \frac{\sin x\cos x - \sinh x\cosh x}{\sin^2 x\cosh^2 x} \leq \frac{\cos x(\sin x - \sinh x)}{\sin^2 x\cosh^2 x} < 0,$$

which implies that f_3 is decreasing. By Lemma 1, $f(x)$ is decreasing. By l'Hôpital rule, $f(0+) = 1$, $f(\pi/4-) = \log\cosh(\pi/4)/\log\sqrt{2} \approx 0.811133 > 2/3$. □

Next, we improve the bounds of (1.3).

Theorem 3. *If $x \in (0, 1)$ and $a = \log\sinh 1/\log\cosh 1 \approx 0.372168$, then*

$$\left(\frac{1}{\cosh x}\right)^{a} < \frac{x}{\sinh x} < \left(\frac{1}{\cosh x}\right)^{\frac{1}{3}} \tag{2.3}$$

with the best possible constants a and $1/3$.