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Periodic solutions for a generalized p-Laplacian equation

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ABSTRACT

The existence and uniqueness of T-periodic solutions for the following boundary value problems with p-Laplacian:

$$(\phi_p(x'))' + f(t,x') + g(t,x) = e(t), \qquad x(0) = x(T), \qquad x'(0) = x'(T)$$

are investigated, where $\phi_p(u) = \mid u\mid^{p-2} u$ with $p > 1$ and $f,\ g,\ e$ are continuous and are T-periodic in t with $f(t,0) = 0$. Using coincidence degree theory, some existence and uniqueness results are presented.

1. Introduction

We consider the solvability of the following periodic boundary value problem:

$$(\phi_p(x'))' + f(t,\ x') + g(t,\ x) = e(t) \tag{1}$$

$$x(0) = x(T), \qquad x'(0) = x'(T), \tag{2}$$

where $\phi_p(u) = |u|^{p-2}u$ with $p > 1$ and $f,\ g$ and e are continuous functions and are T-periodic in t with $f(t,0) = 0$.

When $p = 2$, Eq. (1) reduces to the following second-order forced Rayleigh equation:

$$x'' + f(t,x') + g(t,x) = e(t). \tag{3}$$

The existence and uniqueness of periodic solutions of (1) and (3) have been an important research focus for the study of dynamic behaviors of nonlinear second-order differential equations. See, for example, the research papers [1–9] and the references therein. Recently, Wang [7] has obtained the following results:

Theorem A. *Consider the following p-Laplacian equation:*

$$(\phi_p(x'))' + Cx' + g(t,\ x) = e(t) \tag{4}$$

where C is a constant and $\int_0^T e(t)dt = 0$. Assume that there exists $d \geq 0$ such that:

(H$_1$) $[g(t,u_1) - g(t,u_2)](u_1 - u_2) < 0\ \forall\, u_1, u_2, u_1 \neq u_2$, *and* $t \in \mathbb{R}$.
(H$_2$) $xg(t,x) < 0\ \forall\, |x| > d$, *and* $t \in \mathbb{R}$.

Then (4) has a unique T-periodic solution.

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Theorem B. *Assume that $g(t, x) = g(x)$ and (H_1) holds. Then (4) has a unique T-periodic solution if and only if $0 \in g(\mathbb{R})$.*

Remark 1. For the case of $p = 2$, related results were obtained by Li and Huang [2] and Wang and Shao [6].

In this work, we discuss the existence and uniqueness of T-periodic solutions of (1)–(2) under some general conditions. The main result of this work is the following:

Theorem 1. *In the problem (1)–(2), assume that:*

(A_1) *There exists $d^* > 0$ such that*

$$x(g(t, x) - e(t)) < 0 \quad \forall t \in \mathbb{R} \quad \text{and} \quad |x| \geq d^*.$$

(A_2) *There exist nonnegative constants m_1, m_2 and $\alpha \leq p - 1$ such that*

$$|f(t, u)| \leq m_1|u|^\alpha + m_2 \quad \forall t, u \in \mathbb{R} \quad \text{and} \quad f(t, 0) = 0\, \forall\, t \in \mathbb{R}.$$

Then the problem (1)–(2) has at least one solution. Moreover, if $p \geq 2$ and (H_1) holds, then the problem (1)–(2) has a unique solution.

2. Lemmas

We first introduce some well-known results for p-Laplacian-like operators, which will be used in the proof of Theorem 1. Let $X = C_T^1$ be the space of all C^1-functions which are T-periodic, i.e.,

$$X = C_T^1 = \{x(t) \in C^1(\mathbb{R}, \mathbb{R}) : x(0) = x(T),\ x'(0) = x'(T)\}.$$

The norm of a function $x \in C_T^1$ is defined by

$$\|x\| := \|x\|_\infty + \|x'\|_\infty,$$

where $\|x\|_\infty := \max_{t\in\mathbb{R}} |x(t)|$ and $\|x'\|_\infty := \max_{t\in\mathbb{R}} |x'(t)|$.

Lemma 1 (*Theorem 3.1 [5]*)**.** *Consider the following problem:*

$$(\phi_p(u'))' = h(t, u, u'), \qquad u(0) = u(T), \qquad u'(0) = u'(T), \tag{5}$$

where $\phi_p(u) = |u|^{p-2}u$ with $p > 1$ and h is a Caratheodory function and is T-periodic in t. Let $\Omega = \{x \in C_T^1 : \|x\| < r\}$ for some $r > 0$. Suppose that the following conditions hold:

(i) *For each $\lambda \in (0, 1)$, the problem*

$$(\phi_p(u'))' = \lambda h(t, u, u'), \quad u \in C_T^1 \tag{6}$$

has no solution on $\partial\Omega$.

(ii) *The function $H(a)$ defined by*

$$H(a) := \frac{1}{T}\int_0^T h(t, a, 0)dt$$

satisfies $H(-r)H(r) < 0$.

Then the problem (5) has at least one solution in Ω.

Lemma 2 (*Generalized Bellman's Inequality*)**.** *Consider the following inequality:*

$$|y(t)| \leq C + M\int_0^t |y(s)|^\beta ds \tag{7}$$

where C, M, β are nonnegative constants and $t > 0$. If $\beta \leq 1$, then for $t \in (0, T_0]$, we have $|y(t)| \leq D$, where

$$D = \begin{cases} Ce^{MT_0}, & \text{if } \beta = 1, \\ (C^{1-\beta} + MT_0(1-\beta))^{\frac{1}{1-\beta}}, & \text{if } \beta < 1. \end{cases}$$

Proof of Theorem 1. Let $h(t, x, x') = e(t) - f(t, x') - g(t, x)$. Then (6) reduces to

$$(\phi_p(x'))' + \lambda f(t, x') + \lambda[g(t, x) - e(t)] = 0, \quad \lambda \in (0, 1). \tag{8}$$

We first show that the set of all possible T-periodic solutions of (8) is bounded. Let $x(t)$ be an arbitrary T-periodic solution of (8). Assume that

$$x(t_1) = \max_{t\in[0,T]} x(t), \qquad x(t_2) = \min_{t\in[0,T]} x(t), \quad t_1, t_2 \in [0, T].$$

Then we have

$$x'(t_1) = x'(t_2) = 0, \qquad x''(t_1) \leq 0, \qquad x''(t_2) \geq 0.$$