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Sharpness and generalization of Jordan's inequality and its application

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ABSTRACT

Let $\theta \geq 2$ be a given real number, and $a, b \in \mathbb{R}$ be two parameters, and let

$$Q(x; a, b, \theta) = \frac{2}{\pi} + a\left(\pi^{\theta} - (2x)^{\theta}\right) + b\left(\pi^{\theta} - (2x)^{\theta}\right)^2.$$

We determine the values

$$a = \frac{2\pi^{-\theta-1}}{\theta}, \qquad b = \frac{\left(-\pi^2 + 4 + 4\theta\right)\pi^{-2\theta-1}}{4\theta^2},$$

which provide the best approximation:

$$\frac{\sin x}{x} \approx Q\left(x; \frac{2\pi^{-\theta-1}}{\theta}, \frac{\left(-\pi^2 + 4 + 4\theta\right)\pi^{-2\theta-1}}{4\theta^2}, \theta\right), \quad 0 < x \leq \frac{\pi}{2}.$$

Furthermore, we establish a sharp Jordan's inequality, and then apply it to improve the Yang Le inequality.

1. Introduction and preliminaries

The Jordan's inequality (see [1], p. 33)

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1, \quad 0 < x \leq \frac{\pi}{2} \tag{1.1}$$

has important applications in many areas of pure and applied mathematics. This simple inequality has motivated a large number of research papers concerning its new proofs, various generalizations, sharpness and applications (see [2–20] and the references cited in them).

The following sharp lower and upper bounds for the function $\frac{\sin x}{x}$ were proved in [2,6,8,16,19]:

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2) \quad 0 < x \leq \frac{\pi}{2}. \tag{1.2}$$

Recently, Zhu [20] established new sharp lower and upper bounds for the function $\frac{\sin x}{x}$ as follows:

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{12 - \pi^2}{16\pi^5}(\pi^2 - 4x^2)^2 \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{\pi - 3}{\pi^5}(\pi^2 - 4x^2)^2 \tag{1.3}$$

where $0 < x \leq \pi/2$.

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Two analogues of the inequalities (1.2) and (1.3):

$$\frac{2}{\pi}+\frac{2}{3\pi^4}(\pi^3-8x^3)\le\frac{\sin x}{x}\le\frac{2}{\pi}+\frac{\pi-2}{\pi^4}(\pi^3-8x^3) \tag{1.4}$$

and

$$\frac{2}{\pi}+\frac{1}{2\pi^5}(\pi^4-16x^4)\le\frac{\sin x}{x}\le\frac{2}{\pi}+\frac{\pi-2}{\pi^5}(\pi^4-16x^4) \tag{1.5}$$

were established (see [3,4]).

In view of the inequalities (1.2)–(1.5), we now introduce the approximations family

$$\frac{\sin x}{x}\approx Q(x;a,b,\theta),\quad 0<x\le\frac{\pi}{2}, \tag{1.6}$$

where

$$Q(x;a,b,\theta)=\frac{2}{\pi}+a(\pi^\theta-(2x)^\theta)+b(\pi^\theta-(2x)^\theta)^2, \tag{1.7}$$

and $\theta\ge 2$ is a given real number, and $a,b\in\mathbb{R}$ are parameters.

The first aim of this work is to determine the values

$$a=\frac{2\pi^{-\theta-1}}{\theta},\qquad b=\frac{(-\pi^2+4+4\theta)\pi^{-2\theta-1}}{4\theta^2},$$

which provide the best approximation:

$$\frac{\sin x}{x}\approx Q\left(x;\frac{2\pi^{-\theta-1}}{\theta},\frac{(-\pi^2+4+4\theta)\pi^{-2\theta-1}}{4\theta^2},\theta\right)\quad 0<x\le\frac{\pi}{2}. \tag{1.8}$$

The second aim of this work is to give a unified sharpness and generalization of the above inequalities. By using the result obtained, we improve the well-known Yang Le inequality, which is the third aim of this work.

Before stating and proving the main theorems, we first introduce here two lemmas.

Lemma 1. *For all integers $n\ge 3$ and all real numbers $\theta\ge 2$,*

$$2\left(\frac{4^n(4^n-1)|B_{2n}|}{(2n)!}\right)\theta^2+3\left(\frac{4^n(4^n-1)|B_{2n}|}{(2n)!}-\frac{4^{n-1}(4^{n-1}-1)|B_{2n-2}|}{(2n-2)!}\right)\theta+\frac{4^n(4^n-1)|B_{2n}|}{(2n)!}\ge 0, \tag{1.9}$$

where the B_{2n} are the Bernoulli numbers defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

The equality in (1.9) occurs for $n=3$ and $\theta=2$.

Proof. We first show that the inequality (1.9) is true for $n=3,4$ and $\theta\ge 2$. When $n=3$, inequality (1.9) is

$$\frac{4}{15}(\theta-2)^2+\frac{7}{15}(\theta-2)\ge 0,\quad \theta\ge 2.$$

When $n=4$, inequality (1.9) is

$$\frac{34}{315}(\theta-2)^2+\frac{61}{315}(\theta-2)+\frac{1}{105}\ge 0,\quad \theta\ge 2.$$

Now we are in a position to prove that the inequality (1.9) is true for $n\ge 5$ and $\theta\ge 2$. The inequality (1.9) can be written for $n\ge 5$ and $\theta\ge 2$ as

$$(2\theta^2+3\theta+1)\frac{4^n(4^n-1)|B_{2n}|}{(2n)!}>3\frac{4^{n-1}(4^{n-1}-1)|B_{2n-2}|}{(2n-2)!}\theta. \tag{1.10}$$

It is known [21, p. 805] that

$$\frac{2(2n)!}{(2\pi)^{2n}}<|B_{2n}|<\frac{2(2n)!}{(2\pi)^{2n}(1-2^{1-2n})},\quad n\ge 1. \tag{1.11}$$

Therefore, in order to prove (1.10) it is sufficient to prove that.

$$(2\theta^2+3\theta+1)\frac{4^n(4^n-1)}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}}>3\frac{4^{n-1}(4^{n-1}-1)}{(2n-2)!}\frac{2(2n-2)!}{(2\pi)^{2n-2}(1-2^{3-2n})}\theta,$$