Applied Mathematics Letters 23 (2010) 812–817

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Approximation theorems for certain positive linear operators

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ARTICLE INFO

Article history:
Received 27 April 2009
Received in revised form 28 October 2009
Accepted 22 March 2010

Keywords:
Korovkin approximation
Positive operator
q-Lupaş–Bernstein
ω, q-Bernstein operator
Iterates

ABSTRACT

In this work we prove approximation theorems for certain positive linear operators via Ditzian–Totik moduli $\omega_{2,\phi}(f,\cdot)$ of second order where the step-weights are functions whose squares are concave. The results obtained are applied to the q-Lupaş–Bernstein operators, the ω, q-Bernstein operators and the convergence of the iterates of the q-Bernstein polynomials.

1. Introduction

In recent years, the q-Bernstein polynomials have been investigated intensively and a comprehensive review of the results on q-Bernstein polynomials along with an extensive bibliography on the subject is given in [1,2]. The well-known Korovkin Theorem plays a key role in the study of approximation with positive linear operators, while Bernstein polynomials serve as a leading particular case. Notice that despite being positive linear operators for $0 < q < 1$, the q-Bernstein polynomials, the q-Meyer–König and Zeller operators, the King-type generalization of the q-Bernstein polynomial as well as the ω, q-Bernstein operators do not satisfy the conditions of the Korovkin Theorem. However, they do satisfy the conditions of the Korovkin-type theorem proved by Wang [3] and generalized by Mahmudov [4]. These results guarantee the existence of the limit operator T_∞ for the sequence $\{T_n\}$ which, unlike the situation in the classical case, is not the identity operator. On the other hand, recently Felten [5] proved direct estimates for arbitrary positive linear operators via Ditzian–Totik moduli $\omega_{2,\phi}(f,\delta)$ of second order where the ϕ step-weights are functions whose squares are concave. Unifying and modifying the proofs of the aforementioned three papers we obtain local and global approximation theorems for certain positive linear operators that include some q-parametric operators.

The work is organized as follows. In Section 2 we provide direct estimates for certain positive linear operators. In Section 3 we give the proof of the main result. In Section 4 we apply direct estimates obtained in the previous sections to the q-Lupaş–Bernstein operators, the ω, q-Bernstein operators and the convergence of the iterates of the q-Bernstein polynomials.

2. Approximation results for q-parametric operators

In this work we study positive linear operators of continuous functions which are defined on $[0,1]$. We use the weighted K-functional of second order for $f \in C[0,1]$ defined by

$$K_\phi^2\left(f,\delta^2\right) := \inf_{g' \in AC[0,1]} \left(\|f-g\| + \delta^2 \left\|\phi^2 g''\right\|\right), \quad \delta \geq 0,$$

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doi:10.1016/j.aml.2010.03.016

where $\|\cdot\|$ denotes the uniform norm on $C[0,1]$ and $g' \in AC[0,1]$ means that g is differentiable and g' is absolutely continuous in $[0,1]$. Moreover, the Ditzian–Totik moduli of second and first order are given by

$$\omega_{2,\phi}(f,\delta) := \sup_{|h|\le\delta}\ \sup_{x\pm h\phi(x)\in[0,1]} |f(x-\phi(x)h) - 2f(x) + f(x+\phi(x)h)|,$$

$$\omega_{\phi}(f,\delta) := \sup_{|h|\le\delta}\ \sup_{x\pm(h/2)\phi(x)\in[0,1]} \left|f\left(x+\phi(x)\frac{h}{2}\right) - f\left(x-\phi(x)\frac{h}{2}\right)\right|,$$

$$\overrightarrow{\omega}_{\phi}(f,\delta) := \sup_{|h|\le\delta}\ \sup_{x+h\phi(x)\in[0,1]} |f(x+\phi(x)h) - f(x)|,$$

where $\phi : [0,1] \to R$ is an admissible step-weight function. It is well known that the K-functional $K_{2,\phi}\left(f,\delta^2\right)$ and the Ditzian–Totik modulus $\omega_{2,\phi}(f,\delta)$ are equivalent. Likewise $\omega_{\phi}(f,\delta)$ and $\overrightarrow{\omega}_{\phi}(f,\delta)$ are equivalent; see [6].

Theorem 1. *Assume that $\phi, \psi : [0,1] \to R$ are admissible step-weight functions of the Ditzian–Totik modulus such that ϕ^2 is concave. Suppose that $T_n : C[0,1] \to C[0,1]$ is a sequence of bounded positive linear operators satisfying*

(T1) $T_n(e_0;x) = 1$, $T_n(e_1;x) = a_n x$, $0 < a_n \uparrow 1$ *and*

$$\lim_{n\to\infty} \|T_n(e_2) - T_\infty(e_2)\| = 0.$$

(T2) *$\{T_n(h;x)\}$ is a nonincreasing sequence for any positive nondecreasing convex function h and for any $x \in [0,1]$.*
Then there is a linear positive operator $T_\infty : C[0,1] \to C[0,1]$ such that

$$|(T_n - T_\infty)(f;x)| \le \overrightarrow{\omega}_{\psi}\left(f, \frac{|(1-a_n)x|}{\psi(x)}\right) + C\omega_{2,\phi}\left(f, \sqrt{\left(\frac{T_n\left((t-x)^2;x\right)}{\phi^2(x)} - \frac{T_\infty\left((t-x)^2;x\right)}{2\phi^2_{\max}}\right)}\right) \tag{1}$$

holds true for $x \in [0,1]$ and $f \in C[0,1]$, where C depends on ψ and ϕ and where $\phi^2_{\max} = \max\left\{\phi^2(s) : 0 \le s \le 1\right\}$.

Remark 2. If $\phi = \psi = 1$, Theorem 1 has a local character and it generalizes Wang's and Mahmudov's results. In this case, simplifying the proof of Theorem 1 we get the following estimation:

$$|(T_n - T_\infty)(f;x)| \le \overrightarrow{\omega}(f,(1-a_n)x) + C\omega_2\left(f, \sqrt{(T_n - T_\infty)\left((t-x)^2;x\right)}\right). \tag{2}$$

Corollary 3. *Let T_n and ϕ be given as in Theorem 1. If T_n preserves linear functions then*

$$|(T_n - T_\infty)(f;x)| \le C\omega_{2,\phi}\left(f, \sqrt{\frac{T_n\left(t^2;x\right) - x^2}{\phi^2(x)} - \frac{T_\infty\left(t^2;x\right) - x^2}{2\phi^2_{\max}}}\right) \tag{3}$$

for $x \in [0,1]$ and $f \in C[0,1]$, where the constant C depends only on ϕ.

Corollary 4. *Let T_n and ϕ be given as in Theorem 1. If T_n preserves linear functions then*

$$|T_n(f;x) - f(x)| \le C\omega_{2,\phi}\left(f, \sqrt{\frac{T_n\left(t^2;x\right) - x^2}{\phi^2(x)}}\right) \tag{4}$$

for $x \in [0,1]$ and $f \in C[0,1]$, where the constant C depends only on ϕ.

3. Proofs of the main results

Proof of Theorem 1. In [4] it is proved that under the conditions (T1) and (T2) there exists a linear positive operator $T_\infty : C[0,1] \to C[0,1]$ such that $\lim_{n\to\infty} \|T_n f - T_\infty f\| = 0$ for any $f \in C[0,1]$. So for any positive nondecreasing convex function h and for any $x \in [0,1]$ we have

$$T_n(h;x) \ge T_{n+1}(h;x) \ge \cdots \ge T_\infty(h;x), \tag{5}$$

or in other words $(T_n - T_\infty)(h;x) \ge 0$.

Let $x \in [0,1]$ be fixed and $g' \in AC[0,1]$ be arbitrary. Introduce the following auxiliary functions:

$$g_\pm(t) = \left\|\phi^2 g''\right\| \int_0^t \frac{t-s}{\phi^2(s)}ds + \left\|g'\right\| t + \|g\| \pm g(t).$$