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On the von Neumann–Sion minimax theorem in KKM spaces

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ABSTRACT

In an abstract convex space $(E, D; \Gamma)$, we show that the partial KKM principle is equivalent to a Fan–Browder type fixed point theorem and that this theorem implies generalized forms of the von Neumann–Sion minimax theorem.

1. Introduction

The von Neumann–Sion minimax theorem is fundamental in convex analysis and in game theory. von Neumann [1] proved his theorem for simplexes by reducing the problem to the one-dimensional cases. Sion's generalization [2] was proved by the aid of Helly's theorem and the KKM theorem due to Knaster et al. [3]. In a recent paper, Kindler [4] proved Sion's theorem by applying the one-dimensional KKM theorem (i.e., every interval in $\mathbb{R}$ is connected), the one-dimensional Helly theorem (i.e., any family of pairwise intersecting compact intervals in the real line $\mathbb{R}$ has a nonempty intersection), and Zorn's lemma (or other method).

In a recent work of the author [5], for convex subsets X of a topological vector space E, he showed that a KKM principle implies a Fan–Browder type fixed point theorem and that this theorem implies a generalized form of the Sion minimax theorem.

In the present paper, the procedure in [5] can be generalized and applied to abstract convex spaces recently due to the author. In fact, in an abstract convex space $(E, D; \Gamma)$, he shows that the partial KKM principle is equivalent to a Fan–Browder type fixed point theorem and that this theorem implies generalized forms of the von Neumann–Sion minimax theorem.

2. Abstract convex spaces

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Recall the following in [6–10]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the *Γ-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_N \mid N \in \langle D' \rangle\} \subset E.$$

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A subset X of E is called a *Γ-convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D' \rangle})$ is called a *Γ-convex subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such a case, a subset X of E is said to be *Γ-convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example 2.1. The following are known examples of abstract convex spaces; see [6–10].

(1) The original KKM theorem is for the triple $(\Delta_n, V; \mathrm{co})$, where Δ_n is the standard n-simplex, V the set of its vertices $\{e_i\}_{i=0}^n$, and co: $\langle V \rangle \multimap \Delta_n$ the convex hull operation.
(2) Fan's celebrated KKM lemma is for $(E, D; \mathrm{co})$, where D is a nonempty subset of a topological vector space E.
(3) A *convex space* $(X; \Gamma)$ due to Lassonde.
(4) A C-space $(X; \Gamma)$ due to Horvath.
(5) Hyperconvex metric spaces due to Aronszajn and Panitchpakdi.
(6) Hyperbolic spaces due to Reich and Shafrir.
(7) Any topological semilattice $(X, \leq)$ with path-connected interval introduced by Horvath and Llinares.
(8) A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park.
(9) A *ϕ_A-space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ due to Park.
(10) A space $(H, X; \Gamma)$ due to Kirk and Panyanak, where X is a closed convex subset of a complete $\mathbb{R}$-tree H, and for each $A \in \langle X \rangle$, $\Gamma_A := \mathrm{conv}_H(A)$.
(11) Horvath's convexity spaces.
(12) A $\mathbb{B}$-space due to Briec and Horvath.

Note that each of (2)–(12) has a large number of concrete examples.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \quad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map*.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

In our recent works [6,8,10], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Example 2.2. We give known examples of KKM spaces:

(1) Every G-convex space is a KKM space.
(2) A connected linearly ordered space $(X, \leq)$ can be made into a KKM space.
(3) The extended long line L^* is a KKM space $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$. But L^* is not a G-convex space.
(4) For Horvath's convex space $(X, \mathcal{C})$ with the weak Van de Vel property, the corresponding abstract convex space $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [\![A]\!] = \bigcap\{C \in \mathcal{C} \mid A \subset C\}$ is metrizable for each $A \in \langle X \rangle$.
(5) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space.

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{aligned} \text{Simplex} &\Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space} \\ &\Longrightarrow H\text{-space} \Longrightarrow G\text{-convex space} \Longleftrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space} \\ &\Longrightarrow \text{A space satisfying the partial KKM principle} \\ &\Longrightarrow \text{Abstract convex space.} \end{aligned}$$

3. From the KKM principle to the minimax theorem

For an abstract convex space $(E, D; \Gamma)$, let us consider the following:

Definition. A multimap $T : E \multimap E$ is called a *Fan–Browder map* provided that there exists a companion map $S : E \multimap D$ such that

(a) for each $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; and
(b) $E = \bigcup_{z \in N} \mathrm{Int}\, S^-(z)$ for some finite subset N of D.