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Solution of a hypersingular integral equation in two disjoint intervals

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ABSTRACT

A hypersingular integral equation in two disjoint intervals is solved by using the solution of Cauchy type singular integral equation in disjoint intervals. Also a direct function theoretic method is used to determine the solution of the same hypersingular integral equation in two disjoint intervals. Solutions by both the methods are in good agreement with each other.

1. Introduction

Hypersingular integral equation is considered as an important tool in Applied Mathematics as it finds application in solving a large class of mixed boundary value problems arising in mathematical physics. Particularly the crack problems in fracture mechanics or water wave scattering problems involving barriers, diffraction of electromagnetic waves, aerodynamics problems (cf. [1–3]) can be reduced to hypersingular integral equations in single or disjoint multiple intervals.

A simple hypersingular integral equation is given by

$$Hf = \int_{-1}^{1} \frac{f(t)}{(x-t)^2} \mathrm{d}t = \psi(x), \quad -1 < x < 1, \tag{1.1}$$

where $f \in C^{1,\alpha}(-1, 1)$ and $\psi \in C^{0,\alpha}(-1, 1)(0 < \alpha < 1)$; $C^{n,\alpha}(-1, 1)$ denote the class of functions having Holder continuous derivative of order n with exponent as α.

The hypersingular integral Hf appearing in (1.1) is understood to be equal to Hadamard finite part (cf. [4]) of this divergent integral as given by the relation

$$Hf = \lim_{\epsilon \to 0+} \left[\int_{-1}^{x-\epsilon} \frac{f(t)}{(x-t)^2} \mathrm{d}t + \int_{x+\epsilon}^{1} \frac{f(t)}{(x-t)^2} \mathrm{d}t - \frac{f(x+\epsilon) + f(x-\epsilon)}{\epsilon} \right]. \tag{1.2}$$

Eq. (1.1) has been solved by Martin [4], Chakrabarti and Mandal [5] by utilizing the known solution of Cauchy type singular integral equation of first kind.

Recently, Chakrabarti [6] has developed a direct function theoretic method to determine the solution of (1.1).

In the present paper we have considered for solution the following hypersingular integral equation in two disjoint intervals $G \equiv (-1, -k) \cup (k, 1)$,

$$\frac{1}{\pi} \int_{-1}^{-k} \frac{f(t)}{(x-t)^2} \mathrm{d}t + \frac{1}{\pi} \int_{k}^{1} \frac{f(t)}{(x-t)^2} \mathrm{d}t = \psi(x), \quad x \in G. \tag{1.3}$$

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Here $f(x) \in C^{1,\alpha}(G)$ and $\psi(x) \in C^{0,\alpha}(G)$, $0 < \alpha < 1$ with

$$f(\pm 1) = f(\pm k) = 0 \tag{1.4}$$

and the integral is defined in the sense of Hadamard finite part as described in Eq. (1.2).

We have used two different methods to solve Eq. (1.3). In the first method we have utilized the solution of the following aerofoil equation which is Cauchy type singular integral equation in two disjoint intervals (cf [7])

$$\frac{1}{\pi}\int_{-1}^{-k} \frac{\phi(t)}{x-t}\mathrm{d}t + \frac{1}{\pi}\int_{k}^{1} \frac{\phi(t)}{x-t}\mathrm{d}t = h(x), \quad x \in G.$$

In the second method we have applied direct function theoretic method as described by Chakrabarti [5] to obtain the closed form solution of integral equation (1.3). Solution obtained by both the methods are in good agreement with each other.

In the following sections we give detailed analysis for solution of integral equation (1.3).

2. The detailed analysis

In this section we proceed to solve the hypersingular integral equation (1.3) under condition (1.4) by two different methods.

Method-I

Noting condition (1.4), the hypersingular integral equation (1.3) can be written equivalently as

$$\frac{1}{\pi}\int_{-1}^{-k} \frac{f'(t)}{x-t}\mathrm{d}t + \frac{1}{\pi}\int_{k}^{1} \frac{f'(t)}{x-t}\mathrm{d}t = -\psi(x), \quad x \in G. \tag{2.1}$$

This is the well-known aerofoil equation with Cauchy type singularity which was solved by Tricomi [7] and the solution is given by

$$f'(x) = \begin{cases} \dfrac{1}{\pi R(x)}\left[C_1 + C_2 x + \Psi(x)\right], & x \in (-1, -k), \\ -\dfrac{1}{\pi R(x)}\left[C_1 + C_2 x + \Psi(x)\right], & x \in (k, 1), \end{cases} \tag{2.2}$$

where,

$$\Psi(x) = \int_{-1}^{-k} \frac{\psi(t)R(t)}{x-t}\mathrm{d}t - \int_{k}^{1} \frac{\psi(t)R(t)}{x-t}\mathrm{d}t,$$

C_1, C_2 are two arbitrary constants and

$$R(x) = \{(1-x^2)(x^2-k^2)\}^{\frac{1}{2}}. \tag{2.3}$$

Integrating (2.2) with respect to x gives

$$f(x) = \begin{cases} \dfrac{1}{\pi}\left[\displaystyle\int_{-1}^{x} \frac{1}{R(u)}\left(C_1 + C_2 u + \Psi(u)\right)\mathrm{d}u\right] + F_1, & x \in (-1, -k), \\ \dfrac{1}{\pi}\left[\displaystyle\int_{x}^{1} \frac{1}{R(u)}\left(C_1 + C_2 u + \Psi(u)\right)\mathrm{d}u\right] + F_2, & x \in (k, 1), \end{cases} \tag{2.4}$$

where F_1, F_2 are another two arbitrary constants.

Now, we make an observation that if $f(x)$ has to satisfy the end conditions $f(\pm 1) = 0$ as given in relation (1.4), we must have

$$F_2 = 0, \qquad F_1 = 0. \tag{2.5}$$

Also, the conditions $f(\pm k) = 0$ yields

$$\int_{k}^{1} \frac{1}{R(u)}\left(C_1 + C_2 u + \Psi(u)\right)\mathrm{d}u = 0, \tag{2.6}$$

$$\int_{-1}^{-k} \frac{1}{R(u)}\left(C_1 + C_2 u + \Psi(u)\right)\mathrm{d}u = 0. \tag{2.7}$$

Solving Eqs. (2.6) and (2.7) we get the constants C_1, C_2 which are given by

$$C_1 = \frac{P(k)}{F(q)}, \qquad C_2 = 0, \tag{2.8}$$

where,

$$P(k) = \int_{k}^{1} \frac{\mathrm{d}u}{R(u)} \int_{k}^{1} \frac{tR(t)}{u^2-t^2}\left(\psi(-t) + \psi(t)\right)\mathrm{d}t,$$