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C^1 monotone cubic Hermite interpolant

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ABSTRACT

Constraining an interpolation to be shape preserving is a well established technique for modelling scientific data. Many techniques express the constraint variables in terms of abstract quantities that are difficult to relate to either physical values or the geometric properties of the interpolant. In this paper, we construct a piecewise monotonic interpolant where the degrees of freedom are expressed in terms of the weights of the rational Bézier cubic interpolant.

1. Introduction

There exist many interpolating methods which derive sufficient conditions to ensure the resulting interpolant preserves some geometric aspect of the data, for example, monotonicity. All develop constraints based on parameters of the interpolating polynomial which either has no correspondence with the data or the natural geometry of the interpolant. For example, Fritsch and Butland [1], Fritsch and Carlson [2] and Higham [3] used piecewise cubic Hermite polynomials to interpolate monotone data (increasing data with positive derivatives). Although the cubic Hermite schemes interpolate data and derivatives, they did not preserve the shape of monotone data. The authors in [1–3] proposed different remedies of the problem, for example, the schemes. [2,4] first identified the interval in which monotonicity was lost by ordinary cubic Hermite interpolant then modified the derivatives to obtain the required results. The scheme of [3] Higham inserts extra knots to preserve monotonicity. The scheme proposed by Schumaker [5] was economical due to piecewise quadratic polynomial interpolant and preserved monotonicity. However, its drawback is that it is of degree two only. Gregory and Delbourgo [4] introduced a family of rational quadratic functions with quadratic denominator. They exploited the free parameters of the rational quadratic function to preserve the shape of monotone data. Hussain and Hussain [6] used a rational cubic function to preserve the shape of monotone curve data, again by exploiting the free parameters of the function. Hussain and Sarfraz [7] used a rational cubic function with four free parameters. Two parameters are the constraints for the shape preserving monotone data while two parameters are free for the user to refine the shape of monotone curve. Hussain et al. [8] used a rational cubic function and imposed the shape preserving constraints on the functions one free parameter.

The aim of this paper is to develop a monotonicity preserving curve interpolant based on rational cubic Bézier basis functions where the sufficient conditions are expressed in terms of the weights. We first derive the sufficient conditions for a rational cubic Bézier univariate function to preserve monotonicity. This is then readily extended to vector valued data.

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2. Basics

Given two end points, P_0 and P_3 with $P_3 > P_0$, and two end derivatives, D_0 and D_3, construct a monotonically increasing rational Hermite interpolant. Expressed as a rational cubic Bézier, we have

$$P(u) = \frac{p(u)}{\omega(u)} = \frac{\sum_{i=0}^{3} b_{i,3}(u)\,\omega_i\,P_i}{\sum_{i=0}^{3} b_{i,3}(u)\,\omega_i},$$

where $b_{i,3}(u) =^3 C_i(1-u)^{3-i}u^i$ are the Bernstein basis functions and $P_i i = 0, \ldots, 3$ are the control points of the Bézier curve. Let the input data be the end points and end derivatives, i.e. $P_0 = P(0),\ P_3 = P(1);\ D_0 = \dot{P}(0)$ & $D_3 = \dot{P}(1)$. Then we require the conditions on the weights, $\omega_i\ i = 0, \ldots, 3$ such that $\dot{R}(u) > 0$.

3. Analysis

The sufficient conditions on the weights of the rational cubic Bézier interpolant to preserve monotonicity is

$$\frac{\omega_0\,\omega_3}{\omega_1\,\omega_2} > 3. \quad \square \tag{1}$$

Proof

Writing $\omega(u)\,P(u) = p(u)$ and differentiating with respect to u gives

$$\dot{\omega}(u)\,P(u) + \omega(u)\,\dot{P}(u) = \dot{p}(u).$$

Rearranging gives

$$\dot{P}(u) = \frac{1}{\omega(u)}\left[\dot{p}(u) - \dot{\omega}(u)\,Pu\right] = \frac{1}{\omega(u)^2}\left[\omega(u)\,\dot{p}(u) - \dot{\omega}(u)\,p(u)\right].$$

We note that

$$\dot{P}(0) = \frac{3\omega_1}{\omega_0}\left(P_1 - P_0\right)$$
$$\dot{P}(1) = \frac{3\omega_2}{\omega_3}\left(P_3 - P_2\right).$$

Thus for monotonicity we require

$$S(u) = \left[\omega(u)\,\dot{P}(u) - \dot{\omega}(u)\,p(u)\right] = \left[S_0(u) - S_1(u)\right] > 0$$

since $\omega(u)^2 > 0$. Now

$$S_0(u) = 3\sum_{i=0}^{5} b_{i,5}(u)\mathcal{P}_{0,i} \quad \text{and} \quad S_1(u) = 3\sum_{i=0}^{5} b_{i,5}(u)\mathcal{P}_{1,i},$$

where

$$\mathcal{P}_{0,0} = \omega_0\left(\omega_1 P_1 - \omega_0 P_0\right)$$
$$\mathcal{P}_{0,1} = \left[\frac{3}{5}\omega_1\left(\omega_1 P_1 - \omega_0 P_0\right) + \frac{2}{5}\omega_0\left(\omega_2 P_2 - \omega_1 P_1\right)\right]$$
$$\mathcal{P}_{0,2} = \left[\frac{1}{10}\omega_0\left(\omega_3 P_3 - \omega_2 P_2\right) + \frac{6}{10}\omega_1\left(\omega_2 P_2 - \omega_1 P_1\right) + \frac{3}{10}\omega_2\left(\omega_1 P_1 - \omega_0 P_0\right)\right]$$
$$\mathcal{P}_{0,3} = \left[\frac{3}{10}\omega_1\left(\omega_3 P_3 - \omega_2 P_2\right) + \frac{6}{10}\omega_2\left(\omega_2 P_2 - \omega_1 P_1\right) + \frac{1}{10}\omega_3\left(\omega_1 P_1 - \omega_0 P_0\right)\right]$$
$$\mathcal{P}_{0,4} = \left[\frac{3}{5}\omega_2\left(\omega_3 P_3 - \omega_2 P_2\right) + \frac{2}{5}\omega_3\left(\omega_2 P_2 - \omega_1 P_1\right)\right]$$
$$\mathcal{P}_{0,5} = \omega_3\left(\omega_3 P_3 - \omega_2 P_2\right)$$