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Free vibrations for some Koiter shells of revolution

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Abstract

The asymptotic behaviour of the smallest eigenvalue in linear Koiter shell problems is studied, as the thickness parameter tends to zero. In particular, three types of shells of revolution are considered. A result concerning the ratio between the bending and the total elastic energy is also provided, by using the general theory detailed in [L. Beirão da Veiga, C. Lovadina, An interpolation theory approach to Shell eigenvalue problems (submitted for publication); L. Beirão da Veiga, C. Lovadina, Asymptotics of shell eigenvalue problems, C.R. Acad. Sci. Paris 9 (2006) 707–710].

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1. Introduction and problem description

In considering the free vibrations of shells using the Koiter model (see [8,6,5], for instance), one is led to study the following *eigenvalue* problem in variational form:

$$\begin{cases} \text{Find } (\mathbf{u}_t, \lambda_t) \in V \times \mathbf{R} \text{ such that} \\ t a^m(\mathbf{u}_t, \mathbf{v}) + t^3 a^b(\mathbf{u}_t, \mathbf{v}) = \lambda_t m_t(\mathbf{u}_t, \mathbf{v}) \quad \forall \mathbf{v} \in V \\ \|\mathbf{u}_t\|_0 = 1. \end{cases} \tag{1}$$

Above, t is the shell thickness parameter and V is the space of admissible displacements, incorporating also the kinematic boundary conditions. The bilinear forms $a^m(\cdot,\cdot)$ and $a^b(\cdot,\cdot)$ are independent of t and are associated with the membrane and bending energy, respectively. Finally, $m_t(\cdot,\cdot)$ is the mass bilinear form. We notice that for an eigenvalue λ_t, the corresponding shell vibration frequency is given by $\omega_t = \sqrt{\lambda_t}$.

In this work we are interested in the *smallest* eigenvalue of problem (1), still denoted by λ_t, and in particular we focus on the asymptotic behaviour of the function $t \to \lambda_t$, as $t \to 0^+$. We will also consider the percentage of the

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elastic energy stored in the bending part. Accordingly, for $(\mathbf{u}_t, \lambda_t) \in V \times \mathbf{R}$ solution of (1), we define the function $R(t, \mathbf{u}_t)$ as

$$R(t, \mathbf{u}_t) := \frac{t^3 a^b(\mathbf{u}_t, \mathbf{u}_t)}{\lambda_t}. \tag{2}$$

We examine a set of shells of revolution, whose mid-surfaces are all defined as follows. Let $I \subset \mathbf{R}$ be a bounded closed interval, and let $f : I \to \mathbf{R}^+$ be a regular function. The shell mid-surface is parametrised by means of the mapping

$$\phi : \Omega = I \times [0, 2\pi] \longrightarrow \mathbf{R}^3; \qquad \phi(\xi^1, \xi^2) = (\xi^1, f(\xi^1)\sin\xi^2,\ f(\xi^1)\cos\xi^2). \tag{3}$$

In particular, we study the following shells, which cover the three fundamental types of mid-surface geometry.

$$\text{Parabolic cylinder:} \qquad f''(\xi^1) = 0 \qquad \forall \xi^1 \in I \tag{4}$$

$$\text{Elliptic cylinder:} \qquad f''(\xi^1) < 0 \qquad \forall \xi^1 \in I \tag{5}$$

$$\text{Hyperbolic cylinder:} \qquad f''(\xi^1) > 0 \qquad \forall \xi^1 \in I. \tag{6}$$

For all the shells, we impose clamped boundary conditions at both ends $(\xi^1, \xi^2) \in \partial I \times [0, 2\pi]$. Accordingly, the space of admissible displacements is

$$V = [H_0^1(\Omega)]^2 \times H_0^2(\Omega). \tag{7}$$

We do not need now to explicitly describe the bilinear forms: it is sufficient to recall that:

1. The bilinear forms $a^m(\cdot,\cdot)$ and $a^b(\cdot,\cdot)$ are symmetric and continuous on V.
2. The sum $a^m(\cdot,\cdot) + a^b(\cdot,\cdot)$ is coercive on V.
3. The symmetric and positive-definite mass bilinear form $m_t(\cdot,\cdot)$ satisfies

$$m_t(\mathbf{v}, \mathbf{v}) \sim t\|\mathbf{v}\|_0^2 . \tag{8}$$

We now introduce the following definition (cf. [2]).

Definition 1.1. We say that the eigenvalue problem (1) is of *order* α if

$$\alpha = \inf\left\{\beta : t^\beta \lambda_t^{-1} \in L^\infty(0, 1)\right\} . \tag{9}$$

Remark 1.1. Definition 1.1 means that if the eigenvalue problem is of order α, then α is the "best" exponent in order to have $\lambda_t \sim t^\alpha$. Furthermore, it is easily seen that if the eigenvalue problem (1) is of order α, then $0 \le \alpha \le 2$.

Remark 1.2. In [2,3] a different scaling has been employed for the right-hand side of problem (1). More precisely, the term $\lambda_t m_t(\mathbf{u}_t, \mathbf{v})$ is there replaced by a term of the type $\lambda_t^*(\mathbf{u}_t, \mathbf{v})_0$, where λ_t^* denotes the corresponding eigenvalue. As a consequence of (8), we have $\lambda_t \sim t^{-1}\lambda_t^*$. Accordingly, the problem order α^* is given by $\alpha^* = \alpha + 1$. This shift should be taken into account when comparing the results of the present note with those given in [2,3].

2. Asymptotic behaviour of λ_t and of $R(t, \mathbf{u}_t)$

We first notice that for all the shells under consideration $a^m(\cdot,\cdot)$ defines a norm on V. Indeed, using the clamped boundary conditions, it is easy to see that $a^m(\mathbf{v}, \mathbf{v}) = 0$ if and only if $\mathbf{v} = \mathbf{0}$. We set $H := [L^2(\Omega)]^3$ and W as the completion of V with the norm $a^m(\mathbf{v}, \mathbf{v})^{1/2} := \|\mathbf{v}\|_W$. Therefore, we have the dense inclusion $V \subseteq W$, which implies $W' \subseteq V'$ densely. We have the following result, whose proof involves the interpolation theory (see [4,9], for instance) and can be found in [2].

Theorem 2.1. *Suppose that* $a^m(\mathbf{v}, \mathbf{v}) = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$. *The order* α *of the eigenvalue problem* (1) *is given by*

$$\alpha = \inf\left\{2\theta : H \subseteq (W', V')_{\theta,1}\right\} = \inf\left\{2\theta : (V, W)_{1-\theta,2} \subseteq H\right\}. \tag{10}$$