Applied Mathematics Letters 25 (2012) 884–889

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Stabilization of mixture of two rigid solids modeling temperature and porosity

Margareth S. Alves [a], Jaime E. Muñoz Rivera [b], Mauricio Sepúlveda [c,*], Octavio P. Vera Villagrán [d]

[a] Departamento de Matemática, Universidade Federal de Viçosa-UFV, 36570-000, Viçosa-MG, Brazil
[b] LNCC, Rua Getulio Vargas 333, Quitandinha-Petrópolis, 25651-070, Rio de Janeiro, RJ, Brazil
[c] CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Chile
[d] Departamento de Matemática, Universidad del Bío-Bío, Collao 1202, Casilla 5-C, Concepción, Chile

ARTICLE INFO

Article history:
Received 20 May 2011
Received in revised form 21 October 2011
Accepted 31 October 2011

Keywords:
Analyticity
Exponential stability
C_0-semigroup
Mixture
Temperature
Porosity

ABSTRACT

In this paper we investigate the asymptotic behavior of solutions to the initial boundary value problem for a mixture of two rigid solids modeling temperature and porosity. Our main result is to establish conditions which ensure the analyticity and the exponential stability of the corresponding semigroup.

1. Introduction

This article is concerned with a special case of a linear theory for binary mixtures of porous viscoelastic materials. The theory of viscoelastic mixtures has been investigated by several authors (see for instance, [1–3] and the references therein). In [1] binary mixtures have been considered where the individual components are modeled as porous Kelvin–Voigt viscoelastic materials and the volume fraction of each constituent was considered as an independent kinematical quantity. The authors assumed that the constituents have a common temperature and that every thermodynamical process that takes place in the mixture satisfies the Clausius–Duhem inequality. At the end of that work, they presented as an application the interaction between the temperature field θ and the porosity fields u and w in a homogeneous and isotropic mixture. In this case, and after some considerations, the equations which govern the fields u, w and θ in the absence of body loads are given by the system

$$
\begin{aligned}
&\rho_1 u_{tt} - a_{11}\,\Delta u - a_{12}\,\Delta w - b_{11}\,\Delta u_t - b_{12}\Delta w_t + \alpha\,(u-w) - k_1\,\Delta\theta - \beta_1\,\theta = 0 \quad \text{in } \Omega\times(0,\infty),\\
&\rho_2\, w_{tt} - a_{12}\,\Delta u - a_{22}\,\Delta w - b_{12}\,\Delta u_t - b_{22}\,\Delta w_t - \alpha\,(u-w) - k_2\,\Delta\theta - \beta_2\,\theta = 0 \quad \text{in } \Omega\times(0,\infty),\\
&c\,\theta_t - \kappa\,\Delta\theta + k_1\,\Delta u_t + k_2\,\Delta w_t + \beta_1\,u_t + \beta_2\,w_t = 0 \quad \text{in } \Omega\times(0,\ \infty),
\end{aligned}
\tag{1.1}
$$

* Corresponding author.
E-mail addresses: malves@ufv.br (M.S. Alves), rivera@lncc.br (J.E. Muñoz Rivera), mauricio@ing-mat.udec.cl, masepulveda.cortes@gmail.com (M. Sepúlveda), overa@ubiobio.cl (O.P. Vera Villagrán).

0893-9659/$ – see front matter
doi:10.1016/j.aml.2011.10.044

where Ω is a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. The function $u = u(x,t)$ (and $w = w(x,t)$) represents the fraction field of a constituent and $\theta = \theta(x,t)$ the difference of temperature between the actual state and a reference temperature. We consider the following initial and boundary conditions

$$\begin{aligned} &u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad w(x,0) = w_0, \quad w_t(x,0) = w_1, \quad \theta(x,0) = \theta_0 \quad \text{in } \Omega \\ &u(x,t) = u(x,t) = w(x,t) = w(x,t) = \theta(x,t) = \theta(x,t) = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

We assume that $\rho_1, \rho_2, c, \kappa$, and α are positive constants. Since coupling is considered, we consider $\left(\beta_1^2 + \beta_2^2\right)\left(k_1^2 + k_2^2\right) \neq 0$, but the sign of β_i or k_i $(i = 1, 2)$ does not matter in the analysis. The matrix $A = (a_{ij})$ is symmetric and positive definite and $B = (b_{ij}) \neq 0$ is symmetric and non-negative definite, that is, $a_{11} > 0, a_{11}a_{22} - a_{12}^2 > 0$, $b_{11} \geq 0$ and $b_{11}b_{22} - b_{12}^2 \geq 0$. Our purpose in this work is to investigate the stability of the solutions to the system (1.1)–(1.2). The asymptotic behavior, as $t \to \infty$, of solutions to the equations of linear thermoelasticity has been studied by many authors. Obviously, to get these stability results, we consider several restrictions on the constitutive coefficients. In this sense, this system of equations does not intend to model the general problem. We refer to the book of Liu and Zheng [4] for a general survey on these topics. However, we recall that very few contributions have been addressed to study the time behavior of the solutions of nonclassical elastic theories. In this direction we mention the works [3,5–7]. In [8], the authors treat a similar problem for a one-dimensional mixture modeling temperature and porosity and prove the exponential decay of solutions. We note that we cannot expect that this system always decays in a exponential way. For instance, in case that $\beta_1 + \beta_2 = 0$, $k_1 + k_2 = 0$, $\rho_2\,(a_{11} + a_{12}) = \rho_1\,(a_{12} + a_{22})$ and $b_{11} + b_{12} = b_{12} + b_{22} = 0$ we can obtain solutions of the form $u = w$ and $\theta = 0$. These solutions are undamped and do not decay to zero. These are very particular cases, but we will see that there are some other cases where the solutions decay, but the decay is not so fast to be controlled by an exponential. Our main result is to obtain conditions over the coefficients of the system (1.1) to ensure the exponential stability as well as the analyticity of the semigroup associated with (1.1)–(1.2). We follow the same line of reasoning adopted in the papers [5,6]. This paper is organized as follows. Section 2 outlines briefly the well-posedness of the system is established. In Section 3, we show the exponential stability of the corresponding semigroup provided that certain conditions are guaranteed. In Section 4, we treat the analyticity of the semigroup. In the last Section 5 we show, for some cases, the lack of exponential stability of the semigroup. Throughout this paper C is a generic constant.

2. The existence of the global solution

In this section, we use the semigroup approach to show the well-posedness of the system. We introduce the face space $\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ equipped with the inner product given by

$$\begin{aligned} &\langle (u_1, w_1, v_1, \eta_1, \theta_1), (u_2, w_2, v_2, \eta_2, \theta_2)\rangle_{\mathcal{H}} = a_{11}\,\langle \nabla u_1, \nabla u_2\rangle + a_{22}\,\langle \nabla w_1, \nabla w_2\rangle \\ &\quad + a_{12}\,(\langle \nabla u_1, \nabla w_2\rangle + \langle \nabla w_1, \nabla u_2\rangle) + \alpha\,\langle u_1 - w_1, u_2 - w_2\rangle + \rho_1\,\langle v_1, v_2\rangle + \rho_2\,\langle \eta_1, \eta_2\rangle + c\,\langle \theta_1, \theta_2\rangle \end{aligned}$$

where $\langle u, v\rangle = \int_\Omega u\overline{v}\,dx$, and the induced norms $|\cdot|$ and $\|\cdot\|_{\mathcal{H}}$ which are equivalent to the usual norms in $L^2(\Omega)$ and $\mathcal{H}$, respectively. We also consider the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$

$$\mathcal{A}\begin{pmatrix} u \\ w \\ v \\ \eta \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ \eta \\ \dfrac{1}{\rho_1}\,\Delta(a_{11}u + a_{12}w + b_{11}v + b_{12}\,\eta + k_1\,\theta) - \dfrac{\alpha}{\rho_1}\,(u - w) + \dfrac{\beta_1}{\rho_1}\,\theta \\ \dfrac{1}{\rho_2}\,\Delta(a_{12}u + a_{22}w + b_{12}v + b_{22}\,\eta + k_2\,\theta) + \dfrac{\alpha}{\rho_2}\,(u - w) + \dfrac{\beta_2}{\rho_2}\,\theta \\ \dfrac{1}{c}\,\Delta(\kappa\,\theta - k_1 v - k_2\,\eta) - \dfrac{\beta_1}{c}v - \dfrac{\beta_2}{c}\,\eta \end{pmatrix}$$

whose domain $\mathcal{D}(\mathcal{A})$ is the subspace of $\mathcal{H}$ consisting of vectors (u, v, w, η, θ) such that $v, \eta, \theta \in H_0^1(\Omega)$, $\kappa\,\theta - k_1 v - k_2\,\eta \in H^2(\Omega)$, $a_{11}u + a_{12}w + b_{11}v + b_{12}\,\eta + k_1\,\theta \in H^2(\Omega)$, and $a_{12}u + a_{22}w + b_{12}v + b_{22}\,\eta + k_2\,\theta \in H^2(\Omega)$. The system (1.1)–(1.2) can be rewritten as the following initial value problem $\frac{d}{dt}U(t) = \mathcal{A}U(t)$, $U(0) = U_0$ for all $t > 0$ with $U(t) = (u, w, u_t, w_t, \theta)^T$ and $U_0 = (u_0, w_0, u_1, w_1, \theta_0)^T$, and the T is used to denote the transpose. We can show that the operator $\mathcal{A}$ is densely definite, dissipative, that is, Re $\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leqslant 0$, for all $U \in \mathcal{D}(\mathcal{A})$, and 0 belongs to the resolvent set of $\mathcal{A}$, denoted by $\rho(\mathcal{A})$ (see [6]). Therefore, using the Lumer–Phillips theorem we conclude that the operator $\mathcal{A}$ generates a C_0-semigroup $S_{\mathcal{A}}(t)$ of contractions on the space $\mathcal{H}$. The following theorem follows.

Theorem 2.1. *For any $U_0 \in \mathcal{H}$, there exists a unique solution $U(t) = (u, w, u_t, w_t, \theta)$ of (1.1)–(1.2) satisfying $u, w \in C([0, \infty[: H_0^1(\Omega)) \cap C^1([0, \infty[: L^2(\Omega))$, $\theta \in C([0, \infty[: L^2(\Omega)) \cap L^2(]0, \infty[: H_0^1(\Omega))$. If $U_0 \in \mathcal{D}(\mathcal{A})$ then $u, w \in C^1([0, \infty[: H_0^1(\Omega)) \cap C^2([0, \infty[: L^2(\Omega))$, $\theta \in C([0, \infty[: H_0^1(\Omega)) \cap C^1([0, \infty[: L^2(\Omega))$, and*

$$\begin{aligned} &a_{11}u + a_{12}w + b_{11}\,u_t + b_{12}\,w_t + k_1\,\theta \in C([0, \infty[: H^2(\Omega)) \\ &a_{12}u + a_{22}w + b_{12}\,u_t + b_{22}\,w_t + k_2\,\theta \in C([0, \infty[: H^2(\Omega)) \\ &\kappa\,\theta - k_1\,u_t - k_2\,w_t \in C([0, \infty[: H^2(\Omega)). \end{aligned}$$