Applied Mathematics Letters 24 (2011) 1075–1079

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On Hamiltonian paths in distance graphs

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ARTICLE INFO

Article history:
Received 25 May 2010
Received in revised form 17 January 2011
Accepted 25 January 2011

Keywords:
Distance graph
Circulant graph
Hamiltonian path
Hamiltonian cycle

ABSTRACT

For a finite set $D \subseteq \mathbb{N}$ with $\gcd(D) = 1$, we prove the existence of some $n \in \mathbb{N}$ such that the distance graph P_n^D with vertex set $\{0, 1, \dots, n-1\}$ in which two vertices u and v are adjacent exactly if $|u - v| \in D$, has a Hamiltonian path with endvertices 0 and $n - 1$. This settles a conjecture posed by Penso et al. [L.D. Penso, D. Rautenbach, J.L. Szwarcfiter, Long cycles and paths in distance graphs, Discrete Math. 310 (2010) 3417–3420].

1. Introduction

For a positive integer $n \in \mathbb{N}$ and a finite set of positive integers $D \subseteq \mathbb{N}$, the finite, undirected, and simple *distance graph* P_n^D has vertex set $V(P_n^D) = \{0, 1, \dots, n-1\}$, and two vertices u and v of P_n^D are adjacent exactly if $|u - v| \in D$. Infinite distance graphs and especially their colourings were first studied by Eggleton et al. [1,2] and most research on distance graphs focussed on their colourings. See [3–9] for results on distance graphs.

Distance graphs generalize the very well-studied class of *circulant graphs* [10–13]. In fact, circulant graphs coincide exactly with the regular distance graphs [14]. Circulant graphs have been proposed for numerous network applications and many of their properties such as connectivity and diameter [15,10–12], cycle and path structure [16–18], and isomorphism testing and recognition [19,20], have been studied in great detail.

In [21,14,22,23], some fundamental results concerning circulant graphs are extended to the more general class of distance graphs. One such result, which is obtained by combining the work of Boesch and Tindell [15], Burkard and Sandholzer [24], and Garfinkel [25], states that a circulant graph – that is, a regular distance graph P_n^D – has a Hamiltonian cycle if and only if it is connected if and only if the greatest common divisor of the integers in $\{n\} \cup D$ is 1. The main result of [23] extends this to distance graphs: a distance graph P_n^D with $|D| \geq 2$ has a cycle containing all but a bounded number of vertices if and only if it has a component containing all but a bounded number of vertices if and only if the greatest common divisor $\gcd(D)$ of the integers in D is 1. Next to such almost-Hamiltonian cycles, the authors of [23] study almost-Hamiltonian paths and pose the following conjecture.

Conjecture 1 (*Cf. Conjecture 6 From [23]*)**.** *For a finite set $D \subseteq \mathbb{N}$ with $|D| \geq 2$ and $\gcd(D) = 1$, there is some $n \in \mathbb{N}$ such that $n \geq 2$ and P_n^D has a Hamiltonian path with endvertices 0 and $n - 1$.*

Our goal in the present paper is to prove this conjecture.

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doi:10.1016/j.aml.2011.01.025

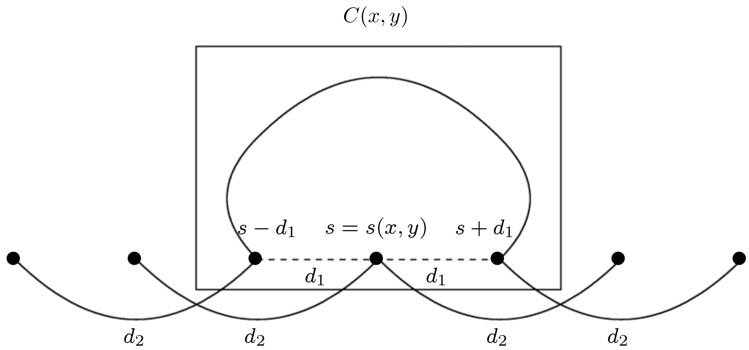

Fig. 1. A respected cell $C(x, y)$.

2. Results

We prepare the proof of our main result with a series of lemmas. The first lemma describes local building blocks, which will later be combined to form long cycles and paths. Note that the second half of the following lemma corresponds to the case $|D| = 2$ of Conjecture 1.

Lemma 2. *Let $D = \{d_1, d_2\}$ for two integers $d_1, d_2 \in \mathbb{N}$ with $\gcd(D) = 1$.*
(i) *$P^D_{d_1+d_2}$ is a cycle.*
(ii) *$P^D_{d_1+d_2+1}$ has two Hamiltonian paths P_{d_1} and P_{d_2} with endvertices 0 and $d_1 + d_2$ such that $E(P^D_{d_1+d_2+1}) = E(P_{d_1}) \cup E(P_{d_2})$.*

Proof. (i) Since $P^D_{d_1+d_2}$ is 2-regular, it is a circulant graph [14]. Since $\gcd(D \cup \{d_1 + d_2\}) = 1$, $P^D_{d_1+d_2}$ has a Hamiltonian cycle [24]. Altogether, $P^D_{d_1+d_2}$ is a cycle.
(ii) If P_{d_1} arises from the cycle $P^D_{d_1+d_2}$ by replacing the edge $\{0, d_1\}$ with the edge $\{d_1, d_1 + d_2\}$ and P_{d_2} arises from the cycle $P^D_{d_1+d_2}$ by replacing the edge $\{0, d_2\}$ with the edge $\{d_2, d_1 + d_2\}$, the desired statement follows. $\square$

A main ingredient of our proof of Conjecture 1 is a partition of the vertex set of the distance graph into so-called cells, which will be defined below. Within each cell, we use the cycle from Lemma 2(i) and stitch these cycles together in order to obtain a long cycle.

Let $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$. If $d_1, d_2 \in D$ with $d_1 < d_2$ and $n = (d_1 + d_2)l$ for some $l \in \mathbb{N}$, then the vertex set $\{0, 1, \ldots, n - 1\}$ of the distance graph P^D_n is the disjoint union of (x, y)-*cells* $C(x, y)$ with

$$C(x, y) = (d_1 + d_2)x + y + \gcd(d_1, d_2) \cdot \left\{0, 1, \ldots, \frac{d_1 + d_2}{\gcd(d_1, d_2)} - 1\right\}$$

for $x \in \{0, 1, \ldots, l - 1\}$ and $y \in \{0, 1, \ldots, \gcd(d_1, d_2) - 1\}$ where $g \cdot A = \{ga \mid a \in A\}$. In each cell, we designate a special vertex $s(x, y) \in C(x, y)$ with

$$\begin{aligned} s(x, y) &= (d_1 + d_2)x + y + \gcd(d_1, d_2) \cdot \left(\frac{d_2}{\gcd(d_1, d_2)} - 1\right) \\ &= (d_1 + d_2)x + y + (d_2 - \gcd(d_1, d_2)). \end{aligned}$$

Note that $s(x, y) = \min(C(x, y)) + d_2 - \gcd(d_1, d_2) = \max(C(x, y)) - d_1$. Furthermore, note that, by Lemma 2(i), the edges uv of P^D_n with $|u - v| \in \{d_1, d_2\}$ and $u, v \in C(x, y)$ define a cycle with vertex set $C(x, y)$.

A subgraph H of P^D_n is said to *respect a cell* $C(x, y) \subseteq V(P^D_n)$ with $1 \leq x \leq l - 2$ if H contains

- all $|C(x, y)| - 2$ edges between vertices u and v in $C(x, y) \setminus \{s(x, y)\}$ with $|u - v| \in \{d_1, d_2\}$,
- the edges $\{s(x, y), s(x, y) + d_2\}$ and $\{s(x, y) + d_1, s(x, y) + d_1 + d_2\}$ between $C_{x,y}$ and $C_{x+1,y}$, and
- the edges $\{s(x, y) - d_1 - d_2, s(x, y) - d_1\}$ and $\{s(x, y) - d_2, s(x, y)\}$ between $C_{x-1,y}$ and $C_{x,y}$.

Note that $s(x, y) + d_1 = \max(C(x, y))$, $s(x, y) + d_1 + d_2 = s(x + 1, y)$, and $s(x, y) - d_1 - d_2 = s(x - 1, y)$. Fig. 1 illustrates all $|C(x, y)| + 2$ edges specified above.

Note that whenever we speak of a cell $C(x, y)$ or of a vertex $s(x, y)$, the two distinct integers d_1 and d_2 are clear from the context.

We will make frequent use of an observation due to Burkard and Sandholzer [24]: if $e = \{a, b\} \in E(H_1)$ and $f = e + d = \{a + d, b + d\} \in E(H_2)$ for two vertex-disjoint graphs H_1 and H_2 with $V(H_1), V(H_2) \subseteq \mathbb{N}_0$, then

$$((E(H_1) \cup E(H_2)) \setminus \{\{a, b\}, \{a + d, b + d\}\}) \cup \{\{a, a + d\}, \{b, b + d\}\}$$

is the edge set of a graph H with vertex set $V(H_1) \cup V(H_2)$ such that every vertex has the same degree in H as in $H_1 \cup H_2$. If H_1 and H_2 are cycles, then H is also a cycle. We say that H arises by *gluing* H_1 *and* H_2 *at* (e, f).

If G is a graph with $V(G) \subseteq \mathbb{Z}$, $c \in \mathbb{N}$, and $d \in \mathbb{Z}$, then $c \cdot G + d$ denotes the graph such that $u \mapsto c \cdot u + d$ defines an isomorphism between G and $c \cdot G + d$.