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Claw-free cubic graphs with clique-transversal number half of their order[☆]

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ABSTRACT

A *clique-transversal set* D of a graph G is a set of vertices of G such that D meets all cliques of G. The *clique-transversal number*, denoted by $\tau_C(G)$, is the minimum cardinality of a clique-transversal set in G. In 2008, we showed that the clique-transversal number of every claw-free cubic graph is bounded above by half of its order. In this note we characterize claw-free cubic graphs which attain the upper bound.

1. Introduction

All graphs considered here are finite, simple and nonempty. For standard terminology not given here we refer the reader to [1].

Let $G = (V, E)$ be a graph with *vertex set* V and *edge set* E. For a vertex $v \in V$, the *open neighborhood* $N(v)$ of v is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u \mid uv \in E\}$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood of S is $N[S] = \bigcup_{v\in S} N[v]$. The *degree* of v is equal to $|N(v)|$, denoted by $d_G(v)$ or simply $d(v)$. If $d_G(v) = k$ for all $v \in V$, then we call G *k-regular*. In particular, a 3-regular graph is also called a *cubic graph*. For a subset $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$, and we let $N_S(v)$ denote the set of vertices in S that are adjacent to v. As usual, $K_{i,j}$ denotes the complete bipartite graph with classes of cardinality i and j; K_n is the complete graph on n vertices. The graph $K_{1,3}$ is also called a *claw* and K_3 a *triangle*. The graph $K_4 - e$ (with one edge removed from K_4) is called a *diamond*. A graph G is said to be *claw-free* if it does not contain an induced subgraph that is isomorphic to a claw. A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G.

The clique-transversal set in graphs can be regarded as a special case of the transversal set in hypergraph theory [2]. A *clique* C of a graph G is a complete subgraph maximal under inclusion and having at least two vertices. According to this definition, isolated vertices are not considered to be cliques here. A clique of order m of G is called an *m-clique* of G. A set $D \subseteq V$ is called a *clique-transversal set* of G if D meets all cliques of G, i.e., $D \cap V(C) \neq \emptyset$ for any clique C of G. The *clique-transversal number*, denoted by $\tau_C(G)$, is the minimum cardinality of a clique-transversal set of G.

Erdős et al. [3] have proved that the problem of finding a minimum clique-transversal set for a graph is NP-hard. It is therefore of interest to determine bounds on the clique-transversal number of a graph. In [3] the authors showed that every graph of order n has clique-transversal number at most $n - \sqrt{2n} + 3/2$ and they observed that τ_C can be very close to

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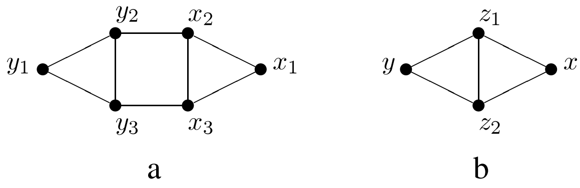

Fig. 1. (a) The double triangle F_1; (b) The diamond F_2.

$n = |V(G)|$, namely $\tau_C = n - o(n)$ can hold. On the other hand, one also observes that τ_C can be very small (possibly $\tau_C = 1$) for general graphs. On the basis of the fact above, it is reasonable to ask how drastically τ_C decreases or increases when some assumptions are imposed on the graph G. From this point of view, Tuza [4] and Andreae [5] established upper bounds on τ_C for chordal graphs. Andreae et al. [6] investigated classes of graphs of order n for which $\tau_C \leq n/2$. In [7] we established sharp lower bounds on τ_C for connected k-regular graphs. Recently, Bacsó and Tuza [8] found the tight upper bound on τ_C for cubic graphs. For other investigations on the clique-transversal sets in graphs, we refer the reader to [9–12].

In [7] we also showed that $3n/8 \leq \tau_C(G) \leq n/2$ for a cubic claw-free graph G of order n. This note gives a characterization of the extremal graphs attaining the upper bound.

2. The main results

Theorem 1 *([7]). For any claw-free cubic graph G of order n where $G \ncong K_4$, $\frac{3}{8}n \leq \tau_C(G) \leq \frac{n}{2}$.*

Lemma 2. *Let G be claw-free cubic graph of order n. If every triangle of G is contained in a diamond, then*

$$\tau_C(G) = \begin{cases} \dfrac{3}{8}n & \text{if } n \equiv 0 (\text{mod } 8); \\ \dfrac{3n+4}{8} & \text{otherwise}. \end{cases}$$

Proof. Since G is claw-free, each vertex of G lies in a triangle. Furthermore, since every triangle of G is contained in a diamond, the graph G can be viewed as a "cycle" on "vertices" where each "vertex" is a copy of a diamond. That is, G is obtained from the disjoint union of k (≥ 2) copies of a diamond by pairing arbitrarily all the vertices of degree 2 in distinct copies and joining all paired vertices of degree 2 so that each vertex of G has degree 3. By the parity of k, we have $n \equiv 0 (\text{mod } 8)$ or $n \equiv 4 (\text{mod } 8)$. By Theorem 1, we have

$$\tau_C(G) \geq \lceil 3n/8 \rceil = \begin{cases} \dfrac{3}{8}n & \text{if } n \equiv 0 (\text{mod } 8); \\ \dfrac{3n+4}{8} & \text{otherwise}. \end{cases}$$

Next we show that the converse inequality holds by induction on k. For $k = 2, 3$, it is easy to check that the assertion is true. Therefore we may assume that $k \geq 4$ for the induction step.

Let A and B be two consecutive diamonds of G. We delete the set $V(A) \cup V(B)$. This yields two vertices of degree 2, say u and v. Note that $uv \notin E(G)$. Add new edge uv and we denote by G' the resulting graph. By the induction hypothesis, the assertion holds for G'. So $\tau_C(G') \leq \lceil 3|V(G')|/8 \rceil = \lceil 3(n-8)/8 \rceil$. Let S' be a minimum clique-transversal set of G'. Obviously, $u \in S'$ or $v \in S'$. It is easy to see that we can choose three vertices x, y, z in $V(A) \cup V(B)$ such that $S' \cup \{x, y, z\}$ is a clique-transversal set of G. Hence $\tau_C(G) \leq |S'| + 3 \leq \lceil 3(n-8)/8 \rceil + 3 \leq \lceil 3n/8 \rceil$. □

For a claw-free cubic graph G, let T be a triangle as an induced subgraph of G and $|N(V(T)) \cap (G - V(T))| = 3$. If $N(V(T)) \cap (G - V(T))$ is an independent set of vertices in G, we call T an *isolated triangle* of G. Otherwise, G must contain a *double triangle* (see Fig. 1(a)) as its induced subgraph.

Theorem 3. *If G is a claw-free cubic graph of order n, then $\tau_C(G) = \frac{n}{2}$ if and only if G is diamond-free.*

Proof. Suppose that G is diamond-free. Since G is cubic and claw-free, each vertex of G lies exactly in one 2-clique and one 3-clique in G. Hence G contains a precise total of $n/2$ 2-cliques and any two 2-cliques in G are disjoint. This implies that $\tau_C(G) \geq n/2$. By Theorem 1, we have $\tau_C(G) = n/2$.

Conversely, suppose that G contains at least one diamond with $\tau_C(G) = n/2$. Choose such a graph G with as few vertices as possible. Then G is connected. If G contains neither isolated triangles nor double triangles, then every triangle of G must be contained in a diamond. By Lemma 2, we have $\tau_C(G) < n/2$, contradicting our assumption. Therefore, G contains isolated triangles or double triangles as its induced subgraphs.

Suppose that G contains double triangles as its induced subgraphs of G. We have the following claim.