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On the properties of a class of log-biharmonic functions

S.A. Khuri

Department of Mathematics and Statistics, American University of Sharjah, United Arab Emirates

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ABSTRACT

The aim of this work is to introduce and investigate a new class of log-biharmonic functions. Certain geometrically motivated properties and results concerning starlikeness, convexity and univalence of elements within this class versus the corresponding harmonic functions are obtained and discussed. In particular, we consider the Goodman–Saff conjecture and prove that the conjecture is true for the logarithms of functions belonging to this class.
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1. Introduction

Complex-valued harmonic functions that are univalent and sense preserving in the unit disk U can be written in the form $f = g + \overline{h}$, where h and g are analytic in U. A continuous complex-valued function $F = u + iv$ in a domain $D \subset C$ is biharmonic if the Laplacian of F is harmonic, that is ΔF is harmonic in D if F satisfies the biharmonic equation $\Delta(\Delta F) = 0$, where $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$. The class of biharmonic functions includes the class of harmonic functions and is a subclass of the class of polyharmonic functions. A continuous complex-valued function $F = u + iv$ in a domain $D \subset C$ is log-biharmonic if $\log F$ is biharmonic, that is the Laplacian of $\log F$ is harmonic. A function G is said to be log-harmonic in D if there is an analytic function a and G is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{G_{\bar{z}}}}{\overline{G}} = a\frac{G_z}{G}.$$

It has been shown that if G is a nonvanishing log-harmonic mapping, then G can be expressed as $G = k\bar{l}$ where k and l are analytic functions in D. It is worth noting that in the latter case the Laplacian of the logarithm of the nonvanishing log-harmonic mapping G is zero, that is $(\log G)_{z\bar{z}} = 0$.

A harmonic function F is locally univalent if the Jacobian of F, J_F,

$$J_F = |F_z|^2 - |F_{\bar{z}}|^2 \neq 0.$$

A function F is orientation preserving if

$$J_F = |F_z|^2 - |F_{\bar{z}}|^2 > 0.$$

We say that a univalent biharmonic (harmonic) function F, with $F(0) = 0$, is starlike if the curve $F(re^{it})$ is starlike with respect to the origin for each $0 < r < 1$. In other words, F is starlike if $\frac{\partial \arg F(re^{it})}{\partial t} = \mathrm{Re}\frac{zF_z - \bar{z}F_{\bar{z}}}{F} > 0$ for $z \neq 0$.

E-mail address: skhoury@aus.edu.

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A univalent biharmonic (harmonic) function, F, with $F(0) = 0$ and $\frac{\partial F(re^{it})}{\partial t} \neq 0$ whenever $0 < r < 1$, is said to be convex if the curve $F(re^{it})$ is convex for each $0 < r < 1$. In other words, F is convex if $\frac{\partial \arg \frac{\partial}{\partial t} F(re^{it})}{\partial t} > 0$ for $z \neq 0$.

The biharmonic equation arises in physical applications including linear elasticity theory and fluid flow. The biharmonic functions, which are closely associated with the biharmonic functions, appear in Stokes flow problems as well as in radar imaging problems. There are several problems involving Stokes flow which arise in engineering and biological transport phenomena. For the various applications of the biharmonic functions see [1–3] and the references within.

Recently, biharmonic and log-biharmonic functions have been studied in a number of papers; see for example [1,2,4,5]. For more details on harmonic mappings and the various definitions introduced see [6–8]. The purpose of this work is to study a class of log-biharmonic functions. Some geometrical properties related to starlikeness, convexity and univalence are examined. Further, we show that the Goodman–Saff conjecture (see [9]) is valid for the logarithm of functions belonging to this class.

2. Properties of the class $\mathcal{LBH}$

In this work, we will consider the following class of functions:

$$\mathcal{LBH} = \{F : F = f(z)h(\bar{z})\, G^{\lambda_1|z|^2+\lambda_2}, \text{ where } G \text{ is a nonvanishing log-harmonic mapping in the unit disk } U \text{ and } G(0) = 1,\ f(z) \text{ and } h(z) \text{ are nonvanishing analytic functions in } U, \lambda_1 \text{ and } \lambda_2\ (\lambda_1^2 + \lambda_2^2 \neq 0) \text{ are constants}\}.$$

It will be shown that this elements in this class are log-biharmonic functions; some geometrical properties related to starlikeness, convexity and univalence for elements in $\mathcal{LBH}$ versus the corresponding harmonic functions and/or log-harmonic functions are derived.

First, define the linear operator $\mathcal{L}$ by

$$\mathcal{L} = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}.$$

The definition leads to the following two properties:

- $\mathcal{L}[\alpha f + \beta g] = \alpha\mathcal{L}[f] + \beta\mathcal{L}[g]$,
- $\mathcal{L}[fg] = f\mathcal{L}[g] + g\mathcal{L}[f]$,

where f, g are C^1 functions and α, β are complex constants.

Theorem 1. *For any $F \in \mathcal{LBH}$, F is log-biharmonic.*

Proof. Let $F = f(z)h(\bar{z})\, G^{\lambda_1|z|^2+\lambda_2} \in \mathcal{LBH}$. Taking the logarithm of both sides, we have

$$\log F = \log f(z) + \log h(\bar{z}) + (\lambda_1|z|^2 + \lambda_2)\log G.$$

Upon differentiating both sides with respect to z and $\bar{z}$, respectively, we get

$$(\log F)_z = \frac{f'(z)}{f(z)} + \lambda_1\bar{z}\log G + (\lambda_1|z|^2 + \lambda_2)(\log G)_z,$$
$$(\log F)_{\bar{z}} = \frac{h'(\bar{z})}{h(\bar{z})} + \lambda_1 z\log G + (\lambda_1|z|^2 + \lambda_2)(\log G)_{\bar{z}}.$$

From the definition of the class $\mathcal{LBH}$ it is required that G is log-harmonic, which means that $(\log G)_{z\bar{z}} = (\log G)_{\bar{z}z} = 0$. Differentiating the latter two equations we have

$$(\log F)_{z\bar{z}} = \lambda_1\log G + \lambda_1\bar{z}(\log G)_{\bar{z}} + \lambda_1 z(\log G)_z,$$
$$(\log F)_{\bar{z}z} = \lambda_1\log G + \lambda_1 z(\log G)_z + \lambda_1\bar{z}(\log G)_{\bar{z}}.$$

We note that $(\log F)_{z\bar{z}} = (\log F)_{\bar{z}z}$. Further differentiation leads to

$$(\log F)_{z\bar{z}z} = \lambda_1(\log G)_z + \lambda_1(\log G)_z + \lambda_1 z(\log G)_{zz}.$$

This latter equation and the fact that $(\log G)_{z\bar{z}} = 0$ yields that $(\log F)_{z\bar{z}z\bar{z}} = 0$. This means that F is log-biharmonic. □

Theorem 2. *Let $F = f(z)h(\bar{z})\, G^{\lambda_1|z|^2+\lambda_2} \in \mathcal{LBH}$. Then:*

a. $\mathcal{L}[\log F] = \mathcal{L}[\log f(z)] + \mathcal{L}[\log h(\bar{z})] + (\lambda_1|z|^2 + \lambda_2)\mathcal{L}[\log G]$,
b. $\mathcal{L}^n[\log F] = \mathcal{L}^n[\log f(z)] + \mathcal{L}^n[\log h(\bar{z})] + (\lambda_1|z|^2 + \lambda_2)\mathcal{L}^n[\log G]$,

where $n \geq 2$ is an integer.

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