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Remarks on some recent metrical common fixed point theorems

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ABSTRACT

In an interesting article, Bouhadjera and Godet-Thobie (2009) [6] introduced notions of subcompatibility and subsequential continuity, and utilized them to prove several common fixed point theorems. The results of the aforementioned article contain flaws, and they are not correct in their present form. But these results can be recovered in two ways by strengthening either of the newly introduced definitions. The results, in their corrected form, still bring about noted improvements over a multitude of relevant fixed point theorems of the existing literature, substantiating the utility of these (two) new notions.

1. Introduction and preliminaries

The first ever attempt to improve the commutativity conditions in common fixed point theorems can be traced back to Sessa [1], who introduced the notion of weakly commuting mappings. In 1986, Jungck [2] further enlarged the class of weakly commuting mappings by introducing the notion of compatible mappings, which has been exploited extensively by the researchers of this domain in the recent past to improve common fixed point theorems. Inspired by the definition of Jungck [2], researchers of this domain have introduced several definitions of compatible-like conditions, such as compatible mappings of type (A), (B), (C) and (P), biased mappings, weakly compatible mappings, occasionally weakly compatible mappings, and some others, whose systematic survey (up to 2001) is available in [3]. For the sake of completeness, we list some relevant weak conditions of commutativity in the following lines.

Definition 1.1 (*[1]*)**.** A pair (f, g) of self mappings of a metric space (X, d) is said to be weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$.

Definition 1.2 (*[2]*)**.** A pair (f, g) of self mappings of a metric space (X, d) is said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \quad \text{for some } t \in X.$$

Clearly, weakly commuting maps are compatible, but the implication is not reversible (see [2]).

Definition 1.3 (*[4]*)**.** A pair (f, g) of self mappings of a nonempty set X is said to be occasionally weakly compatible (in short OWC) if there is some $x \in X$ with $fx = gx$ such that $fgx = gfx$, i.e., there exists at least one coincidence point of the pair at which pair commutes.

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Definition 1.4 (*[5]*)**.** A pair (f, g) of self mappings of a metric space (X, d) is said to be reciprocally continuous iff $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$, for every sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

Clearly, any pair of continuous mappings is reciprocally continuous, but the converse need not be true in general.

Most recently, Bouhadjera and Godet-Thobie [6] weakened the notions of reciprocal continuity and compatibility (at the same time extending the OWC one) by coining the following definitions.

Definition 1.5 (*[6]*)**.** A pair (f, g) of self mappings of a metric space (X, d) is said to be subcompatible iff there exists a sequence $x_n \in X$ such that $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

Definition 1.6 (*[6]*)**.** A pair (f, g) of self mappings of a metric space (X, d) is said to be subsequential continuous iff there exists a sequence $x_n \in X$ such that

$$\lim_{n\to\infty} fgx_n = ft \quad \text{and} \quad \lim_{n\to\infty} gfx_n = gt$$

with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

If f and g are both continuous or reciprocally continuous, then they are obviously subsequentially continuous. But there exists a pair of subsequentially continuous mappings which are neither continuous nor reciprocally continuous. For such examples and other related details, refer to [6].

2. Main results

Using newly introduced definitions, Bouhadjera and Godet-Thobie [6] established the following results, besides proving some other results too.

Theorem 2.1 (Cf. *[6]*)**.** *Let f, g, h, and k be self maps of a metric space (X, d). If the pairs (f, h) and (g, k) are subcompatible and subsequentially continuous, then*

(a) *f and h have a coincidence point,*
(b) *g and k have a coincidence point.*
Let $\psi : \Re_+^6 \to \Re$ be an upper semi-continuous function which satisfies the condition $(\psi 1) : \psi(u, u, 0, 0, u, u) > 0$ for all $u > 0$.
Further, suppose that the pairs (f, h) and (g, k) satisfy the following condition: for all x and y in X,

$$(\psi 2) : \psi(d(fx, gy), d(hx, ky), d(fx, hx), d(gy, ky), d(hx, gy), d(ky, fx)) \leq 0.$$

(c) *Then f, g, h, and k have a unique common fixed point.*

Theorem 2.2 (Cf. *[6]*)**.** *Let f, g, h, and k be self maps of a metric space (X, d). If the pairs (f, h) and (g, k) are subcompatible and subsequentially continuous, then*

(a) *f and h have a coincidence point,*
(b) *g and k have a coincidence point.*
Let $\psi' : \Re_+^4 \to \Re$ be an upper semi-continuous function which satisfies the condition $(\psi_1') : \psi'(u, u, u, u) > 0$ for all $u > 0$.
Further, suppose that the pairs (f, h) and (g, k) satisfy the following inequality: for all x and y in X,

$$(\psi_2') : \psi'(d(fx, gy), d(hx, ky), d(hx, gy), d(ky, fx)) \leq 0.$$

(c) *Then f, g, h, and k have a unique common fixed point.*

Bouhadjera and Godet-Thobie [6] proved above theorems for six and four distances, respectively, which are the generalizations of some results contained in [7]. In addition to above two results, the authors also proved some Gregus-type common fixed point theorems which improve the results due to Djoudi and Nisse [8] and Pathak et al. [9]. Unfortunately, Theorems 2.1 and 2.2 are not true in their present form, but can be recovered by replacing subcompatible pairs with compatible pairs or replacing subsequentially continuous pairs with reciprocally continuous pairs. The error crept in due to the fact that the sequences satisfying the requirements of newly introduced definitions need not be the same as utilized in the proofs of the article [6]. To substantiate this viewpoint, we furnish the following example, which disproves Theorems 2.1 and 2.2.

Example 2.1. Consider $X = [0, \infty)$ endowed with the natural metric d, and define $f, g : X \to X$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1 + x, & \text{if } x \in (0, 1] \\ 2x - 1, & \text{if } x \in (1, \infty) \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 - x, & \text{if } x \in [0, 1) \\ 3x - 2, & \text{if } x \in [1, \infty). \end{cases}$$