Applied Mathematics Letters 22 (2009) 1386–1389

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On Ricceri's conjecture for a class of nonlinear eigenvalue problems ☆

Xianling Fan *

Department of Mathematics, Lanzhou City University, Lanzhou, 730070, PR China
Department of Mathematics, Lanzhou University, Lanzhou, 730000, PR China

ARTICLE INFO

Article history:
Received 29 September 2008
Received in revised form 19 March 2009
Accepted 30 March 2009

Keywords:
Nonlinear eigenvalue problem
Elliptic equation
Dirichlet problem
Lipschitz condition
Ricceri conjecture

ABSTRACT

In this work we give a negative answer to a conjecture proposed by B. Ricceri in the reference [B. Ricceri, A remark on a class of nonlinear eigenvalue problems, Nonlinear Anal. 69 (2008) 2964–2967] for a class of nonlinear elliptic eigenvalue problems.

The aim of the present work is to give a negative answer to the conjecture proposed by Ricceri [1] for a class of nonlinear elliptic eigenvalue problems.

Ricceri [1] dealt with the following eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda f}$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\lambda \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(0) = 0$.

Since $f(0) = 0$, 0 is a solution of $(P_{\lambda f})$ for each $\lambda \in \mathbb{R}$. Define

$$\Lambda_f = \{\lambda > 0 \mid (P_{\lambda f}) \text{ has at least one non-zero classical solution}\}.$$

For fixed $L > 0$, define

$$\mathcal{B}_L = \{f \mid f : \mathbb{R} \to \mathbb{R} \text{ is Lipschitzian with Lipschitz constant } L \text{ and } f(0) = 0\},$$

$$\mathcal{C}_L = \left\{ f \mid f \in \mathcal{B}_L \text{ and } \sup_{\xi \in \mathbb{R}} \int_0^{\xi} f(t)\mathrm{d}t = 0 \right\}.$$

Let λ_1 be the first eigenvalue for the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

As usual, we adopt the convention $\inf \emptyset = +\infty$.

☆ The research was supported by the National Natural Science Foundation of China (10671084).

* Corresponding address: Department of Mathematics, Lanzhou City University, Lanzhou, 730070, PR China.
E-mail addresses: fanxl@lzu.edu.cn, fanxlmath@gmail.com.

0893-9659/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2009.03.012

Sato and Yanagida [2] proved the equality

$$\inf_{f\in\mathcal{B}_L}\inf\Lambda_f=\frac{\lambda_1}{L}. \tag{2}$$

Ricceri [1] proved the inequality

$$\inf_{f\in\mathcal{C}_L}\inf\Lambda_f\geq\frac{3\lambda_1}{L} \tag{3}$$

and proposed the following conjecture:

Conjecture (*[1]*). *For every* $L>0$, *one has*

$$\inf_{f\in\mathcal{C}_L}\inf\Lambda_f=\frac{3\lambda_1}{L}. \tag{4}$$

In [3] the author has proved the following result.

Proposition 1 (*[3], Theorem 3*). *For every* $L>0$, *one has that* $\frac{3\lambda_1}{L}\notin\Lambda_f$ *for all* $f\in\mathcal{C}_L$.

As was mentioned in [3], after proving Proposition 1, Ricceri's conjecture is still open. It is proved in [3] that, if we use the Carathéodory function $f(x,u)$ instead of $f(u)$ in the problem $(P_{\lambda f})$, then the corresponding Ricceri-type conjecture is correct. The main result of the present work is the following theorem which gives a negative answer to Ricceri's conjecture.

Theorem 1. *For every* $L>0$, *one has*

$$\inf_{f\in\mathcal{C}_L}\inf\Lambda_f>\frac{3\lambda_1}{L}. \tag{5}$$

The nonlinear eigenvalue problems of the form $(P_{\lambda f})$ have been studied extensively and many interesting results for various f have been obtained. The equality (2) established by Sato and Yanagida [2] for the class $\mathcal{B}_L$ gives the precise value of $\inf_{f\in\mathcal{B}_L}\inf\Lambda_f$, which is a fine result. $\mathcal{C}_L$ is a special subclass of $\mathcal{B}_L$. For example, the function $f(t)=-\sin t$ belongs to $\mathcal{C}_1$. Studying the precise value of $\inf_{f\in\mathcal{C}_L}\inf\Lambda_f$ is interesting. Our Theorem 1 gives a negative answer to the Ricceri's conjecture but does not give the precise value of $\inf_{f\in\mathcal{C}_L}\inf\Lambda_f$. The inequality (5) is valid for any bounded smooth domain Ω; however, perhaps the precise value of $\inf_{f\in\mathcal{C}_L}\inf\Lambda_f$ is dependent on Ω.

Note that, by the regularity results (see e.g. [4]), when $f\in\mathcal{B}_L$, every weak solution of $(P_{\lambda f})$ is a classical solution. Because $\frac{f}{L}\in\mathcal{C}_1$ for $f\in\mathcal{C}_L$, we only need to prove Theorem 1 in the case of $L=1$.

The following result revealing the properties of the elements in $\mathcal{C}_1$ is proved in [3].

Lemma 1 (*[3], Lemma 3*). *For every* $f\in\mathcal{C}_1$, *one has that* $\frac{f(\xi)}{\xi}\leq\frac{1}{3}$ *for all* $\xi\in\mathbb{R}\setminus\{0\}$, *and the equation* $\frac{f(\xi)}{\xi}=\frac{1}{3}$ *has at most one positive solution and one negative solution.*

It follows from Lemma 1 that, for every $f\in\mathcal{C}_1$ there holds

$$f(\xi)\xi\leq\frac{1}{3}\xi^2,\quad\forall\xi\in\mathbb{R}. \tag{6}$$

Remark 1. We point out that inequality (3), proved by Ricceri [1], can be obtained from (6). Indeed, in order to prove (3) with $L=1$ we can argue by contradiction. Assume that there exists $f\in\mathcal{C}_1$ and $\lambda<3\lambda_1$ such that $\lambda\in\Lambda_f$. Then $(P_{\lambda f})$ has a non-zero solution u. Thus, by (6) and $\lambda<3\lambda_1$, we have that

$$\int_\Omega|\nabla u|^2\,dx=\lambda\int_\Omega f(u)u\,dx\leq\frac{\lambda}{3}\int_\Omega|u|^2\,dx<\lambda_1\int_\Omega|u|^2\,dx,$$

which contradicts the fact that λ_1 is the first eigenvalue of (1).

The following lemma can be proved immediately from the definition of $\mathcal{C}_1$ and hence the proof is omitted here.

Lemma 2. *Let* $f\in\mathcal{C}_1$ *and* $t>0$. *Set* $g(\xi)=\frac{f(t\xi)}{t}$ *for* $\xi\in\mathbb{R}$. *Then* $g\in\mathcal{C}_1$.

Proof of Theorem 1. We only prove Theorem 1 in the case of $L=1$. Arguing by contradiction, assume that

$$\inf_{f\in\mathcal{C}_1}\inf\Lambda_f=3\lambda_1.$$

Then there are $\{f_n\}\subset\mathcal{C}_1$, $\{\mu_n\}\subset(0,+\infty)$ and $\{u_n\}\subset H_0^1(\Omega)\setminus\{0\}$ such that $\mu_n\to3\lambda_1$ as $n\to\infty$ and

$$-\Delta u_n=\mu_nf_n(u_n)\quad\text{in }\Omega\text{ for }n=1,2,\ldots.$$