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On the spectral radii of weighted double stars☆

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ABSTRACT

The spectra of weighted graphs are given attention by some authors because the graphs in the design of networks and electronic circuits are usually weighted. In this short paper, we completely determine the spectra of weighted double stars. We also give the weighted double star that achieves the maximal spectral radius.

1. Introduction

To find the spectrum of a graph is an elementary problem in spectral theory. However, the spectra of weighted graphs are less studied. There are few results. One in [1] gives the spectrum of weighted star $K_{1,n-1}$ which is also the maximum weighted tree on n vertices. Another result in [2] proves that the weighted double star $S_{n-3,1}^{W^*}$ (see Fig. 1) has the second largest spectral radius of all the weighted trees on n vertices. In this short paper, the spectra of weighted double stars are completely determined, from which we give the weighted double star that achieves the maximal spectral radius. First we give some definitions. The terms and symbols not defined here can be found in [2,3].

Let G be a weighted simple graph of order n with weight set $W = \{w_i | w_j \geq w_{j+1} > 0,\ j = 1, 2, \ldots, m-1;\ i = 1, 2, \ldots, m\}$. The vertex set of G is labeled as $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. A bijection $f : E \longleftrightarrow W$ is called the *weighted function* of G. For convenience, we denote $f(v_iv_j) = w_{ij} \in W$ for any $v_iv_j \in E$. Let $A = (a_{ij})_{n\times n}$ be the adjacency matrix of G. The *weighted-adjacency matrix* of G with respect to f is defined by $A_f(W) = (a_{ij}(W))_{n\times n}$, for short A_f since W is generally given, where

$$a_{ij}(W) = \begin{cases} w_{ij} & \text{if } a_{ij} = 1 \\ 0 & \text{if } a_{ij} = 0. \end{cases}$$

In this situation, $G_f = (V, E, W, f)$ is said to be a *weighted graph* with weighted-adjacency matrix A_f. Clearly, G can be viewed as G_f if $w_{ij} \equiv 1$ for any $w_{ij} \in W$. The *spectrum* of G_f is defined as the spectrum of A_f, and the *spectral radius* of G_f is denoted by $\rho(A_f)$.

Given G and W, different weighted functions give different weighted-adjacency matrices and so give different weighted graphs, all of which we call the *weighted graphs of G with respect to W*, and denote this family by

$$\mathcal{G}(G, W) = \{G(V, E, W, f) \mid f : E \longleftrightarrow W\}.$$

Let $\mathcal{F}$ be the set consisting of all the bijections from E to W. Then $|\mathcal{G}(G, W)| \leq |\mathcal{F}| \leq m!$.

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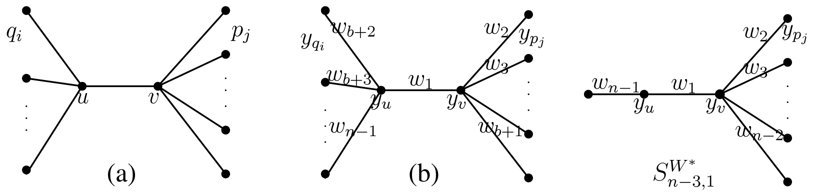

Fig. 1. The double star $S(a, b)$, maximal weighted double star $S^W(a, b)$ and maximum weighted double star $S^{W^*}_{n-3,1}$.

Given G and W, let $\rho = \max\{\rho(G_f) \mid G_f \in \mathcal{G}(G, W)\}$. There exists $f^* \in \mathcal{F}$ such that $\rho = \rho(G_{f^*})$, where G_{f^*} is called the *maximal weighted graph* in $\mathcal{G}(G, W)$. Given W, let $\mathbb{G}$ be the given family of the graphs such that, for any $G \in \mathbb{G}$, $|E(G)| = |W|$. Set

$$\rho_{\max} = \max\{\rho(G_{f^*}) \mid G_{f^*} \text{ is the maximal in } \mathcal{G}(G, W),\ \forall G \in \mathbb{G}\}.$$

There exists $G \in \mathbb{G}$ such that $\rho_{\max} = \rho(G_{f^*_{\max}})$, where $G_{f^*_{\max}} \in \mathcal{G}(G, W)$ is called the *maximum weighted graph* in $\mathbb{G}$.

A *double star* is a tree with exactly two vertices whose degree are greater than one which are called the *centers* of the double star. Denote by $S(a, b)$ the double star of order $n = a+b+2$ with two central vertices u and v such that $|N(u) \setminus v| = a$ and $|N(v) \setminus u| = b$, which is shown in Fig. 1(a). In this short paper, we give the spectra of weighted double stars in Section 2, and find the maximal and maximum weighted double stars in Section 3.

2. The spectra of weighted double stars

Let $S^w(a, b)$ be the weighted double star with weight set $W = \{w_i | w_j \geq w_{j+1} > 0,\ j = 1, 2, \ldots, n-2;\ i = 1, 2, \ldots, n-1\}$, which is shown in Fig. 1(b), where its vertices are labeled as $v, p_1, \ldots, p_b, q_1, \ldots, q_a, u$. Let $\lambda \neq 0$ be an eigenvalue of $S^w(a, b)$ with respect to an eigenvector $\boldsymbol{y} = (y_v, y_{p_1}, \ldots, y_{p_b}, y_{q_1}, \ldots, y_{q_a}, y_u)^T$, where y_u and y_v correspond to centers u and v. Then $A_f\boldsymbol{y} = \lambda\boldsymbol{y}$, which gives according to Fig. 1(b)

$$\begin{cases} \lambda y_{p_j} = w_{k_{1+j}} y_v \quad j = 1, 2, \ldots, b \\ \lambda y_{q_i} = w_{k_{b+1+i}} y_u \quad i = 1, 2, \ldots, a \\ \lambda y_v = \sum\limits_{1 \leq j \leq b} w_{k_{1+j}} y_{p_j} + w_{k_1} y_u \\ \lambda y_u = \sum\limits_{1 \leq i \leq a} w_{k_{b+1+i}} y_{q_i} + w_{k_1} y_v \end{cases} \tag{1}$$

where $w_{k_1}, w_{k_2}, \ldots, w_{k_{n-1}}$ is a permutation of $w_1, w_2, \ldots, w_{n-1}$.

Lemma 2.1. *Let $\boldsymbol{y} = (y_v, y_{p_1}, \ldots, y_{p_b}, y_{q_1}, \ldots, y_{q_a}, y_u)^T$ be an eigenvector of $S^w(a, b)$ corresponding to a non-zero eigenvalue λ. Then $y_u \neq 0$ and $y_v \neq 0$.*

Proof. On the contrary suppose that $y_v = 0$. From (1), we have $y_{p_j} = 0$ for $j = 1, 2, \ldots, b$, and so $w_{k_1} y_u = 0$. Since $w_{k_1} > 0$, we have $y_u = 0$, which gives $y_{q_i} = 0$ ($i = 1, 2, \ldots, a$) as above. It is a contradiction since $\boldsymbol{y} \neq \boldsymbol{0}$. Similarly $y_u \neq 0$. □

Lemma 2.2. *A weighted double star $S^w(a, b)$ has eigenvalue* 0 *with multiplicity $n-4$, and has only four non-zero eigenvalues.*

Proof. It is easy to verify that $\operatorname{rank}(A_f) = 4$. Thus the root space of $A_f X = 0$ has dimension $n-4$ which is exactly the number of linear independent eigenvectors of A_f corresponding to the eigenvalue 0. Hence A_f has eigenvalue 0 with multiplicity $n-4$ since A_f is real symmetric. Consequently, A_f has four non-zero eigenvalues. □

In the following, we will use the Eq. (1) to solve the non-zero eigenvalues of $S^w(a, b)$. Without loss of generality we may assume $\boldsymbol{y} = (y_v, y_{p_1}, \ldots, y_{p_b}, y_{q_1}, \ldots, y_{q_a}, y_u)^T$ is a unit vector. From the first two equations of (1), we have

$$\begin{cases} \lambda y_{p_j}^2 = w_{k_{1+j}} y_{p_j} y_v, \quad j = 1, 2, \ldots, b \\ \lambda y_{q_i}^2 = w_{k_{b+1+i}} y_{q_i} y_u, \quad i = 1, 2, \ldots, a. \end{cases} \tag{2}$$

Multiplying respectively y_v and y_u to the last two equations of (1), we have

$$\begin{cases} \lambda y_v^2 = \sum\limits_{1 \leq j \leq b} w_{k_{1+j}} y_{p_j} y_v + w_{k_1} y_u y_v \\ \lambda y_u^2 = \sum\limits_{1 \leq i \leq a} w_{k_{b+1+i}} y_{q_i} y_u + w_{k_1} y_u y_v. \end{cases} \tag{3}$$