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A note on a Wiener process with measurement error

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ABSTRACT

In this paper we give a closed form for the determinant and the inverse matrix of the covariance matrix of a Wiener process with measurement error. We will discuss its application in the analysis of degradation data for highly-reliable products.

1. Introduction

The covariance matrix of a Wiener process with measurement error takes the form

$$\mathbf{\Omega}_m = a\mathbf{Q}_m + b\mathbf{I}_m, \quad \mathbf{Q}_m = [\min\{t_i, t_j\}]_{1\le i,j\le m},$$

where $a, b > 0$, $0 = t_0 < t_1 < \cdots < t_m$ and $\mathbf{I}_m$ is an identity matrix of order m. The interest of the study of this matrix $\mathbf{\Omega}_m$ appears to be very important not only from a theoretical viewpoint in combinatorics or linear algebra, but also in applications. For instance, it is useful in the study of asymptotic theory (see [1,2] and the references therein) and in the degradation model of highly-reliable products (see [3,4]).

Finding the determinant and the inverse of the covariance matrix $\mathbf{\Omega}_m$ is usually required in these fields. However, the stochastic process is usually assumed to be observed at an equally-spaced grid. i.e., $\Delta t_i = t_i - t_{i-1} = \Delta t$ for all $i = 1, \ldots, m$. Under the assumption of the equally-spaced sampling scheme, the determinant and the inverse of the covariance matrix $\mathbf{\Omega}_m$ can be easily solved by a second order difference equation under given boundary conditions (see [5]). Hence, it is of great interest to release the assumption to adapt more scientific and engineering work.

In the next section, we provide an explicit closed form of the determinant and the inverse of the covariance matrix $\mathbf{\Omega}_m$. An immediate application of this inversion to the degradation data analysis is given in the final section.

2. Determinant and inverse

To avoid reverse product in the following formula, we define $\prod_{i=n_1}^{n_2} c_i = 1$ for $n_2 < n_1$.

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Theorem 1.

$$|\mathbf{\Omega}_m| = b^m + \sum_{r=1}^{m} a^r b^{m-r} \sum_{1\le i_1<\cdots<i_r\le m} t_{i_1} \prod_{j=1}^{r-1}(t_{i_{j+1}} - t_{i_j}).$$

Proof. We prove by mathematical induction. For $m = 2$, it is easy to see that $|\mathbf{\Omega}_2| = b^2 + ab\,(t_1 + t_2) + a^2 t_1(t_2 - t_1)$. Assume that the result holds for $3 \le m \le l$. For the case $m = l + 1$, subtracting the lth column of this determinant from the $l + 1$th column and performing a similar operation with the rows, we obtain the following relation

$$\begin{aligned}|\mathbf{\Omega}_{l+1}| &= \begin{vmatrix} at_1+b & at_1 & \cdots & at_1 & at_1 & 0\\ at_1 & at_2+b & \cdots & at_2 & at_2 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ at_1 & at_2 & \cdots & at_{l-1}+b & at_{l-1} & 0\\ at_1 & at_2 & \cdots & at_{l-1} & at_l+b & -b\\ 0 & 0 & \cdots & 0 & -b & a(t_{l+1}-t_l)+2b \end{vmatrix}\\ &= (a(t_{l+1}-t_l)+2b)\,|\mathbf{\Omega}_l| - b^2|\mathbf{\Omega}_{l-1}|. \end{aligned} \tag{1}$$

Substituting $|\mathbf{\Omega}_l|$ and $|\mathbf{\Omega}_{l-1}|$ into (1), we obtain

$$\begin{aligned}|\mathbf{\Omega}_{l+1}| &= b^{l+1} + ab^l t_{l+1} + \sum_{r=1}^{l} a^r b^{l+1-r} \sum_{1\le i_1<\cdots<i_r\le l} t_{i_1}\prod_{j=1}^{r-1}(t_{i_{j+1}}-t_{i_j})\\ &\quad + \underbrace{t_{l+1}\sum_{r=1}^{l} a^{r+1}b^{l-r} \sum_{1\le i_1<\cdots<i_r\le l} t_{i_1}\prod_{j=1}^{r-1}(t_{i_{j+1}}-t_{i_j})}_{(2.1)} - ab^l t_l + \underbrace{\sum_{r=1}^{l} a^{r+1}b^{l-r} \sum_{1\le i_1<\cdots<i_r\le l} (-t_l)t_{i_1}\prod_{j=1}^{r-1}(t_{i_{j+1}}-t_{i_j})}_{(2.2)}\\ &\quad + \underbrace{\sum_{r=1}^{l} a^r b^{l+1-r} \sum_{1\le i_1<\cdots<i_r\le l} t_{i_1}\prod_{j=1}^{r-1}(t_{i_{j+1}}-t_{i_j})}_{(2.3)} - \sum_{r=1}^{l-1} a^r b^{l+1-r} \sum_{1\le i_1<\cdots<i_r\le l-1} t_{i_1}\prod_{j=1}^{r-1}(t_{i_{j+1}}-t_{i_j}). \end{aligned} \tag{2}$$

Expanding (2.2) and (2.3), we get

$$\begin{aligned}(2.2) &= \underbrace{-\sum_{r=1}^{l-1} a^{r+1}b^{l-r} \sum_{1\le i_1<\cdots<i_r\le l-1} t_l t_{i_1}\prod_{j=1}^{r-1}(t_{i_{j+1}}-t_{i_j})}_{(3.1)} \underbrace{-a^2b^{l-1}t_l^2}_{(3.2)}\\ &\quad \times \underbrace{-\sum_{r=1}^{l-1} a^{r+2}b^{l-1-r} \sum_{1\le i_1<\cdots<i_r\le l-1} t_l(t_l-t_{i_r})t_{i_1}\prod_{j=1}^{r-1}(t_{i_{j+1}}-t_{i_j})}_{(3.3)} \end{aligned} \tag{3}$$

and

$$(2.3) = \sum_{r=1}^{l-1} a^r b^{l+1-r} \sum_{1\le i_1<\cdots<i_r\le l-1} t_{i_1}\prod_{j=1}^{r-1}(t_{i_{j+1}}-t_{i_j}) + ab^l t_l + \underbrace{\sum_{r=1}^{l-1} a^{r+1}b^{l-r} \sum_{1\le i_1<\cdots<i_r\le l-1} (t_l-t_{i_r})t_{i_1}\prod_{j=1}^{r-1}(t_{i_{j+1}}-t_{i_j})}_{(4.1)}. \tag{4}$$

First, combining (3.1) and (4.1), we obtain

$$(3.1) + (4.1) = -\sum_{r=1}^{l-1} a^{r+1}b^{l-r} \sum_{1\le i_1<\cdots<i_r\le l-1} t_{i_1}t_{i_r}\prod_{j=1}^{r-1}(t_{i_{j+1}}-t_{i_j}). \tag{5}$$

Next, combining (3.2), (3.3) and (5), we get

$$(3.2) + (3.3) + (5) = -\sum_{r=1}^{l} a^{r+1}b^{l-r} \sum_{1\le i_1<\cdots<i_r\le l} t_{i_1}t_{i_r}\prod_{j=1}^{r-1}(t_{i_{j+1}}-t_{i_j}). \tag{6}$$