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Nonlinear fractional cone systems with the Caputo derivative

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ABSTRACT

Nonlinear fractional cone systems involving the Caputo fractional derivative are considered. We establish sufficient conditions for the existence of at least one cone solution to such systems. Sufficient conditions for the unique existence of the cone solution to a nonlinear fractional cone system are given.

1. Introduction

Positive systems are, roughly speaking, systems whose trajectories are entirely contained in the nonnegative orthant $\mathbb{R}^n_+$ whenever the initial state and input are nonnegative. For those who are familiar with the notion of the viability of differential equations this is a special case of the viability. In this case, we can say that for "positive viability", the set with respect to which it is considered is just the orthant $\mathbb{R}^n_+$.

The idea of cone systems was introduced in [1]. By a cone system the author means a system obtained from a positive one by substitution of the positive orthants of states and inputs by suitable arbitrary cones. So cone systems are an extension of positive ones and, similarly to positive systems, can be considered in viability terms: we say that a system is viable with respect to a cone if there exists at least one trajectory that is contained in this cone. Such viability with respect to a cone for the linear case of fractional continuous time systems was considered in [2,3]. Results for nonlinear but scalar cases come from such authors as Zhang [4] and Daftardar-Gejji [5].

In this work we glue together and generalize the aforementioned results. Namely, we take into consideration a nonlinear fractional cone system with parameter u described by the equation

$$\left({}^{C}D_0^q x\right)(t) = f(t, x, u), \qquad x(0) = x_0, \tag{1}$$

where $0 < q < 1$, ${}^{C}D_0^q$ is the standard Caputo fractional derivative starting at $t_0 = 0$, and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and input vectors. Moreover, we assume u to be bounded on $[0, T]$ and f to be continuous with respect to x and u on $[0, T]$. Since in (1) there is a parameter u that can also be treated as a control, we refer the reader to a paper of Ibrahim [6], who dealt with similar fractional systems. We introduce special cones $\mathcal{P}_T$, $\mathcal{U}_T$ and K and give sufficient conditions for a solution to (1) to be viable with respect to $\mathcal{P}_T$.

2. Preliminaries

The uniform formula for a fractional integral with $q \in (0, 1)$ is defined as

$$\left(\mathcal{D}_0^{-q} x\right)(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{x(\tau)}{(t-\tau)^{1-q}} d\tau,$$

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where $x(\cdot)$ is an arbitrary integrable function, $D_0^{-q}x$ is the fractional integral of order $q \in (0, 1)$ on $[0, T]$ and Γ is the well-known gamma function. For an arbitrary real number q, the Riemann–Liouville and Caputo fractional derivatives (left-sided) are defined, respectively, as

$$\left(D_0^q x\right)(t) = \frac{d}{dt}\left(\left(\mathcal{D}_0^{q-1}x\right)(t)\right)$$

and

$$\left({}^C D_0^q x\right)(t) = \mathcal{D}_0^{q-1}\left(\frac{d}{dt}x(t)\right).$$

The Caputo derivative is the preferred fractional derivative for engineers.

The left-sided Riemann–Liouville derivatives and the Caputo derivatives coincide if $x(0) = 0$. In that meaning with the initial condition $x_0 = 0$, all results presented here are also true for systems with dynamics defined by the left-sided Riemann–Liouville derivative.

We will need the following definitions and theorem from [5,7] in the sequel.

Definition 1. A Banach space $\mathcal{X}$ endowed with a closed cone K is *an ordered Banach space* $(\mathcal{B}, K)$ with a partial order $\leq_K$ in $\mathcal{B}$ as follows: $\xi_1 \leq_K \xi_2$ if and only if $\xi_2 - \xi_1 \in K$.

Definition 2. For $\xi_1, \xi_2 \in \mathcal{B}$ *the order interval* is defined as $\langle \xi_1, \xi_2 \rangle = \{\xi \in \mathcal{B} : \xi_1 \leq_K \xi \leq_K \xi_2\}$.

Definition 3. Let $(\mathcal{B}, K)$ be an ordered Banach space. Let $\leq_K$ be the order relation induced by a cone K. Cone K is said to be *a normal cone* if there is a constant α such that $\xi_1, \xi_2 \in K$ and $0 \leq_K \xi_1 \leq_K \xi_2 \Rightarrow \|\xi_1\| \leq \alpha \|\xi_2\|$.

Definition 4. Let $\mathcal{B}$ be an ordered Banach space with $\leq_K$ as the order relation. Let $F : \mathcal{B} \supset D \to \mathcal{B}$ be a map; we say that F is *increasing* (*decreasing*) if $\varphi \leq_K \psi$ for $\varphi, \psi \in D$ implies that $F(\varphi) \leq_K F(\psi)$ ($\varphi \leq_K \psi \Rightarrow F(\varphi) \geq_K F(\psi)$).

Theorem 5. *Let $(\mathcal{B}, K)$ be an ordered Banach space with $\langle \bar{\xi}_1, \bar{\xi}_2 \rangle \subset \mathcal{B}$, and $F : \langle \bar{\xi}_1, \bar{\xi}_2 \rangle \to \langle \bar{\xi}_1, \bar{\xi}_2 \rangle$ an increasing continuous operator. If K is a normal cone and F is completely continuous, then F has a fixed point in $\langle \bar{\xi}_1, \bar{\xi}_2 \rangle$.*

Let

$$R^n_{\geq} = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$$

and $R^n_+ = \{x \in \mathbb{R}^n : x_i > 0, 1 \leq i \leq n\}$.

Let us consider the equation given by (1) together with the initial condition $x(0) = x_0$, where $0 < q < 1$, ${}^C D_0^q$ is the standard Caputo fractional derivative, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and input vectors and

$$f(t, x, u) = \begin{bmatrix} f_1(t, x, u) \\ f_2(t, x, u) \\ \vdots \\ f_n(t, x, u) \end{bmatrix}, \tag{2}$$

where $f_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$.

Let $\mathcal{X}_n = C([0, T], \mathbb{R}^n)$ be the Banach space with the supremum norm. Let $\varphi \in \mathcal{X}_n$; then $\|\varphi\| = \max_{i=1,\ldots,n}\left\{\max_{t\in[0,T]} |\varphi_i(t)|\right\}$, where $\varphi_i \in C([0, T], \mathbb{R})$ is the ith component of φ.

On the basis of [2,8], the following definitions are proposed.

Definition 6. Let

$$P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

be a nonsingular matrix and $p_k = (p_{k1}, \ldots, p_{kn})$ be its kth row, $(k = 1, \ldots, n)$. The set

$$K := \left\{x \in \mathbb{R}^n : \forall k = 1, \ldots, n : p_k x \geq 0\right\}$$

is called *a linear cone generated by the matrix P in $\mathbb{R}^n$*. Moreover,

$$\mathcal{P}_T := \{\varphi \in \mathcal{X}_n : \forall t \in [0, T]\ \varphi(t) \in K\}$$

is called *a linear cone generated by the matrix P in the space $\mathcal{X}_n$*. Similarly, taking the matrix $U \in \mathbb{R}^{m\times m}$ we build the cone $\mathcal{U}_T$ in the space $\mathcal{X}_m$.