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Some discrete Fourier transform pairs associated with the Lipschitz–Lerch Zeta function

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ABSTRACT

It is shown that there exists a companion formula to Srivastava's formula for the Lipschitz–Lerch Zeta function [see H.M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, Math. Proc. Cambridge Philos. Soc. 129 (2000) 77–84] and that together these two results form a discrete Fourier transform pair. This Fourier transform pair makes it possible for other (known or new) results involving the values of various Zeta functions at rational arguments to be easily recovered or deduced in a more general context and in a remarkably unified manner.

1. Introduction and definitions

Srivastava's formula [1, p. 81, Eq. (3.9)] reproduced here in (2.2) below, which provides a relationship between the values of the Lipschitz–Lerch and Hurwitz Zeta functions, has been used in several recent papers (see [2, p. 298, Eq. (39)], [3, p. 821, Eq. (23)] and [4, p. 806, Eq. (18)]). In this note we obtain its companion formula and show that together these two results would form a discrete Fourier transform (DFT) pair. For more details regarding the discrete Fourier transforms, the reader is referred to such standard text on the subject as the book by Weaver [5].

A general Hurwitz–Lerch Zeta function $\Phi(z, s, a)$ defined by [6, p. 121 *et seq.*]

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad (a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0,-1,-2,-3,\ldots\}) \qquad (1.1)$$
$$(s \in \mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1)$$

contains, as its *special* cases, not only the Lipschitz–Lerch Zeta function [6, p. 122, Eq. 2.5(11)]:

$$\phi(\xi,a,s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi(e^{2\pi i\xi}, s, a) \qquad (1.2)$$
$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s)>0 \text{ when } \xi \in \mathbb{R}\setminus\mathbb{Z};\ \Re(s)>1 \text{ when } \xi \in \mathbb{Z}\right)$$

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and the Hurwitz (or generalized) and the Riemann Zeta functions:

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(1, s, a) \quad \text{and} \quad \zeta(s) = \Phi(1, s, 1), \tag{1.3}$$

but also other functions such as the Lerch Zeta function [6, p. 122, Eq. 2.5(11)]:

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi} \Phi(e^{2\pi i\xi}, s, 1) \quad (\xi \in \mathbb{R};\ \Re(s) > 1) \tag{1.4}$$

and the Legendre Chi function $\chi_s(z)$ (see, for instance, [7,8]):

$$\chi_s(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s} = \frac{1}{2^s} z \Phi\left(z^2, s, \frac{1}{2}\right) \quad (|z| \leqq 1;\ \Re(s) > 1)\,. \tag{1.5}$$

Finally, the classical Bernoulli polynomials $B_n(x)$ and the classical Bernoulli numbers B_n are defined by (see, for details, [6, p. 61 *et seq.*]; see also a recent work [9]):

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (|t| < 2\pi) \quad \text{and} \quad B_n := B_n(0)$$
$$(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{1.6}$$

2. The main results and their proofs

We begin by observing that, in what follows, we set an empty sum to be zero and it is assumed that p, r and q are positive integers. Our main results in this section are stated and proved as follows. As indicated above, (2.2) is Srivastava's formula [1], while (2.1), (2.3) and (2.4) are presumably new.

Theorem. *Suppose that s and a are complex numbers, $s \neq 1$ and $a \notin \mathbb{Z}_0^-$. Then $\zeta(s, a)$ and $\phi(\xi, a, s)$ form the following* DFT *pair:*

$$\zeta\left(s, \frac{p+a-1}{q}\right) = \frac{1}{q} \sum_{r=1}^{q} q^s \phi\left(\frac{r}{q}, a, s\right) \exp\left(-\frac{2\pi i(r-1)p}{q}\right)$$
$$(p = 1, \ldots, q) \tag{2.1}$$

and

$$\phi\left(\frac{r}{q}, a, s\right) = \frac{1}{q^s} \sum_{p=1}^{q} \zeta\left(s, \frac{p+a-1}{q}\right) \exp\left(\frac{2\pi i(p-1)r}{q}\right)$$
$$(r = 1, \ldots, q). \tag{2.2}$$

Proof. Assume that $\Re(s) > 1$. We first note that Srivastava [1] gave a simple and elegant proof of (2.2) (see, for details, [1, p. 81]). Our proof of (2.1) requires each of the following results:

(a) *Simpson's Series Multisection Formula* (see, for instance, [10, p. 131]). Let

$$f(z) = \sum_{k=1}^{\infty} a_k z^k$$

and let q be fixed. Then, for any p $(1 \leqq p \leqq q)$, we have

$$q \sum_{k=0}^{\infty} a_{p+qk} z^{p+qk} = \sum_{s=1}^{q} \omega^{-sp} f\left(\omega^s z\right) \quad \left(\omega = \exp\left(\frac{2\pi i}{q}\right)\right). \tag{2.3}$$

(b) *Abel's Theorem* (see [11, p. 148]). Let

$$f(z) = \sum_{k=1}^{\infty} a_k z^k.$$

If the series

$$\sum_{k=1}^{\infty} a_k$$