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An improved stability criterion for a class of neutral differential equations

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ABSTRACT

This work gives an improved criterion for asymptotical stability of a class of neutral differential equations. By introducing a new Lyapunov functional, we avoid the use of the stability assumption on the main operators and derive a novel stability criterion given in terms of a LMI, which is less restricted than that given by Park [J.H. Park, Delay-dependent criterion for asymptotic stability of a class of neutral equations, Appl. Math. Lett. 17 (2004) 1203–1206] and Sun et al. [Y.G. Sun, L. Wang, Note on asymptotic stability of a class of neutral differential equations, Appl. Math. Lett. 19 (2006) 949–953].

1. Introduction

We are concerned with the asymptotic stability of the following neutral differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + px(t-\tau)] = -ax(t) + b\tanh x(t-\sigma), \quad t \geq 0, \tag{1.1}$$

where a, τ and σ are positive constants, $\sigma \geq \tau$, b, p are real numbers, $|p| < 1$. For each solution $x(t)$ of Eq. (1.1), we assume the initial condition

$$x(t) = \phi(t), \quad t \in [-\sigma, 0],\ \phi \in C([-\sigma, 0], R).$$

Recently, the asymptotic stability of neutral differential equation (1.1) has been discussed in [1,4,5]. Using stability assumption on the operator

$$\mathcal{D}(x_t) = x(t) + px(t-\tau) + b\int_{t-\sigma}^{t} \tanh x(s)\mathrm{d}s,$$

Park in [4] has proposed a delay-dependent criterion on σ for asymptotical stability of Eq. (1.1). Lately, constructing an improved Lyapunov function based on the operator

$$\mathcal{D}^*(x_t) = x(t) + px(t-\tau) + \alpha\int_{t-\tau}^{t} x(s)\mathrm{d}s + b\int_{t-\sigma}^{t} \tanh x(s)\mathrm{d}s,$$

Sun et al. in [4] have derived a less conservative stability criterion. However, this condition still depends on the stability of the operator $\mathcal{D}^*$. Our aim is to improve these criteria and to break away from some of the assumptions of previous papers. Compared with the results in [4,5], our result has the following advantages:

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- First, we introduce a new Lyapunov functional and avoid the use of the stability assumption on the operator $\mathcal{D}$ in [4] or $\mathcal{D}^*$ in [5]. Hence, our criterion will be less restricted than [4,5] (see Example 2).
- Second, using the operator

$$\mathcal{D}_1(x_t) = x(t) + px(t-\tau) + \alpha \int_{t-\tau}^{t} x(s)\mathrm{d}s + b \int_{t-\sigma}^{t-\tau} \tanh x(s)\mathrm{d}s,$$

the obtained condition depends not only on τ, but also on $\sigma - \tau$. Thus, our criterion is more effective than [5] (see Example 1).

2. Main result

Throughout this section, the symbol $\star$ represents the elements below the main diagonal of a symmetric matrix. Also, $X > Y$ means $X - Y$ is positive definite.

Before present the main result, we need the following technical lemmas.

Lemma 1 (*Completing the Square*). *Assume that $S \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Then for every $Q \in \mathbb{R}^{n\times n}$,*

$$2\langle Qy, x\rangle \le \langle QS^{-1}Q^T x, x\rangle + \langle Sy, y\rangle, \quad \forall x, y \in \mathbb{R}^n.$$

Lemma 2 ([2]). *For any symmetric positive definite matrix $M \in \mathbb{R}^{n\times n}$, scalar $\sigma \ge 0$ and vector function $w : [0, \sigma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then*

$$\left(\int_0^{\sigma} w(s)\mathrm{d}s\right)^T M \left(\int_0^{\sigma} w(s)\mathrm{d}s\right) \le \sigma \int_0^{\sigma} w^T(s) M w(s)\mathrm{d}s.$$

For a real number α, Eq. (1.1) can be rewritten as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) + px(t-\tau) + \alpha \int_{t-\tau}^{t} x(s)\mathrm{d}s + b\int_{t-\sigma}^{t-\tau} \tanh x(s)\mathrm{d}s\right] = (\alpha - a)x(t) - \alpha x(t-\tau) + b\tanh x(t-\tau), \quad t \ge 0. \tag{2.1}$$

Define the following operators:

$$\mathcal{D}_1(x_t) = x(t) + px(t-\tau) + \alpha \int_{t-\tau}^{t} x(s)\mathrm{d}s + b \int_{t-\sigma}^{t-\tau} \tanh x(s)\mathrm{d}s, \qquad \mathcal{D}_2(x_t) = x(t) + px(t-\tau).$$

Now, we have the following theorem.

Theorem 1. *The zero solution of Eq. (1.1) is uniformly asymptotically stable if there exist a constant α with $0 < |\alpha| < 1$ and the positive scalars $\beta, \gamma, \eta, \theta$ such that the following linear matrix inequality holds:*

$$\Omega = \begin{pmatrix} \Omega_{11} & p(\alpha - a) - \alpha & b & \alpha - a & b(\alpha - a) \\ \star & -\beta - 2p\alpha & pb & -\alpha & -b\alpha \\ \star & \star & \theta(\sigma - \tau)^2 - \eta & b & b^2 \\ \star & \star & \star & -\gamma & 0 \\ \star & \star & \star & \star & -\theta \end{pmatrix} < 0, \tag{2.2}$$

where $\Omega_{11} = 2(\alpha - a) + \beta + \gamma\tau^2 + \eta$.

Proof. Since $\Omega < 0$, there is a number $\delta > 0$ such that

$$\Omega_1 = \begin{pmatrix} \Omega_{11} & p(\alpha - a) - \alpha & b & \alpha - a & b(\alpha - a) \\ \star & -\beta - 2p\alpha & pb & -\alpha & -b\alpha \\ \star & \star & \theta(\sigma - \tau)^2 - \eta + \delta & b & b^2 \\ \star & \star & \star & -\gamma & 0 \\ \star & \star & \star & \star & -\theta \end{pmatrix} < 0. \tag{2.3}$$

Consider the following Lyapunov functional:

$$V = V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7,$$

where $V_1 = \mathcal{D}_1^T(x_t)\mathcal{D}_1(x_t)$, $V_2 = \beta\int_{t-\tau}^{t} x^2(s)\mathrm{d}s$, $V_3 = \gamma\tau\int_{t-\tau}^{t}(\tau - t + s)(\alpha x(s))^2\mathrm{d}s$, $V_4 = \theta(\sigma - \tau)\int_{t-\sigma}^{t-\tau}(s - t + \sigma)\tanh^2 x(s)\mathrm{d}s$, $V_5 = \eta\int_{t-\tau}^{t}\tanh^2 x(s)\mathrm{d}s$, $V_6 = \delta\int_{t-\sigma}^{t-\tau}\tanh^2 x(s)\mathrm{d}s$ $V_7 = \epsilon\mathcal{D}_2^2(x_t)$, ϵ is a positive number that will be chosen later.