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# Boundary value problem for a coupled system of nonlinear fractional differential equations

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ABSTRACT

In this work we discuss a boundary value problem for a coupled differential system of fractional order. The differential operator is taken in the Riemann–Liouville sense and the nonlinear term depends on the fractional derivative of an unknown function. By means of Schauder fixed-point theorem, an existence result for the solution is obtained. Our analysis relies on the reduction of the problem considered to the equivalent system of Fredholm integral equations.



## 1. Introduction

We consider a boundary value problem of the coupled system

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)), & 0 < t < 1, \\ D^{\beta}v(t) = g(t, u(t), D^{\nu}u(t)), & 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha, \beta < 2, \mu, \nu > 0, \alpha - \nu \geq 1, \beta - \mu \geq 1, f, g : [0, 1] \times R \times R \to R$ are given functions and $D$ is the standard Riemann–Liouville differentiation.

Fractional differential equation can describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, electromagnetic, etc. There are a large number of papers dealing with the solvability of nonlinear fractional differential equations. The papers [3–7] considered boundary value problems for fractional differential equations. In [7] the authors investigated the existence and multiplicity of positive solutions for a Dirichlet-type problem of the nonlinear fractional differential equation

$$\begin{aligned} &D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ &u(0) = u(1) = 0, \end{aligned}$$

where $1 < \alpha \leq 2$ is a real number, $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous and $D^{\alpha}_{0+}$ is the fractional derivative in the sense of Riemann–Liouville. Because Riemann–Liouville differentiation is not suitable for non-zero boundary values, Zhang, by means of the method of Laplace transforms and fixed-point theorems on a cone, discussed the existence of solutions for nonlinear fractional differential equations with Caputo's derivative and non-zero boundary values in [5,6] respectively. The

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study of a coupled differential system of fractional order is also very significant because this kind of system can often occur in applications [9–11]. In [8], Bai established the existence of a positive solution to a singular coupled system of fractional order.

It is worth mentioning that the nonlinear term in the papers [5–8] is independent of the fractional derivative of the unknown function. But the opposite case is more difficult and complicated and this work attempts to deal exactly with this case. The plan of our work is as follows. In the next section, we prepare some material needed to prove our result. The last section is devoted to the existence of solutions for system (1.1).

## 2. Preliminaries

For completeness, in this section, we recall some definitions and fundamental facts of fractional calculus theory, which can be found in [1,2].

**Definition 2.1.** The fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} \mathrm{d}s,$$

provided that the integral exists.

**Definition 2.2.** The fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to R$ is given by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integral part of number $\alpha$, provided that the right side is pointwise defined on $(0, \infty)$.

**Remark 2.1.** The following properties are useful for our discussion: $I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t), D^{\alpha} I^{\alpha} f(t) = f(t), \alpha > 0, \beta > 0, f \in L(0, 1)$; $I^{\alpha} D^{\alpha} f(t) = f(t), 0 < \alpha < 1, f(t) \in C[0, 1]$ and $D^{\alpha} f(t) \in C(0, 1) \cap L(0, 1)$; $I^{\alpha} : C[0, 1] \to C[0, 1], \alpha > 0$.

## 3. Main results

In this section, we impose growth conditions on $f$ and $g$ which allow us to apply the Schauder fixed-point theorem to establish an existence result for solutions for problem (1.1).

Let $I = [0, 1]$ and $C(I)$ be the space of all continuous real functions defined on $I$.

First of all, we present the Green's function for system (1.1).

**Lemma 3.1.** *Let $\varphi(t) \in C(I)$ be a given function and $1 < \alpha < 2$; then the unique solution of*

$$\begin{cases} D^{\alpha} u(t) = \varphi(t), \quad 0 < t < 1, \\ \varphi(0) = \varphi(1) = 0, \end{cases}$$

*is $u(t) = \int_0^1 G_1(t, s)\varphi(s)\mathrm{d}s$, where*

$$G_1(t, s) = \begin{cases} \dfrac{(t-s)^{\alpha-1} - [t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & s \le t, \\ \dfrac{-[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & t \le s. \end{cases}$$

For the proof of this lemma, we refer the reader to the proof of Lemma 2.3 in [7].

Let

$$G_2(t, s) = \begin{cases} \dfrac{(t-s)^{\beta-1} - [t(1-s)]^{\beta-1}}{\Gamma(\beta)}, & s \le t, \\ \dfrac{-[t(1-s)]^{\beta-1}}{\Gamma(\beta)}, & t \le s \end{cases}$$

we call $(G_1, G_2)$ the Green's functions of the boundary value problem (1.1).

We define the space $X = \{u(t) \mid u(t) \in C(I) \text{ and } D^{\nu} u(t) \in C(I)\}$ endowed with the norm $\|u\|_X = \max_{t\in I} |u(t)| + \max_{t\in I} |D^{\nu} u(t)|$.

**Lemma 3.2.** *$(X, \|\cdot\|_X)$ is a Banach space.*