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The convergence rate for a semilinear parabolic equation with a critical exponent

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ABSTRACT

We study solutions to the Cauchy problem for a semilinear parabolic equation with a nonlinearity which is critical in the sense of Joseph and Lundgren and establish the rate of convergence to regular steady states. In the critical case, this rate contains a logarithmic term which does not appear in the supercritical case.

1. Introduction and result

We consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, & x \in \mathbb{R}^N,\ t \in (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, $p > 1$, Δ denotes the Laplacian operator with respect to x, and the function u_0 is continuous in $\mathbb{R}^N$ and decays to zero as $|x| \to \infty$.

With respect to positive classical steady states of (1.1), there is a critical exponent

$$p_c := \begin{cases} \infty & \text{for } N \le 10, \\ \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{for } N \ge 11, \end{cases}$$

which satisfies $p_c > p_s := \frac{N+2}{N-2}$ for $N \ge 11$ and was found by Joseph and Lundgren (see [1]). More precisely, in the case of $p > p_s$ there is a family of positive radial steady states φ_α, $\alpha > 0$, satisfying

$$\begin{cases} \varphi_{\alpha,rr} + \dfrac{N-1}{r}\varphi_{\alpha,r} + \varphi_\alpha^p = 0, & r > 0, \\ \varphi_\alpha(0) = \alpha, \quad \varphi_{\alpha,r}(0) = 0, \end{cases} \tag{1.2}$$

where $r := |x|$ (see [2–4]). Defining $\varphi_{-\alpha}(r) := -\varphi_\alpha(r)$ and $\varphi_0(r) \equiv 0$, the set $\{\varphi_\alpha \mid \alpha \in \mathbb{R}\}$ is a continuum of radial steady states. Moreover, for $p_s \le p < p_c$ each pair of positive steady states of (1.1) intersect each other. But for $p \ge p_c$ these steady states are strictly ordered such that $\varphi_\alpha(|x|)$ is strictly increasing in α for any $x \in \mathbb{R}^N$ (see [5]) and

$$\lim_{\alpha\to 0} \varphi_\alpha(|x|) = 0, \quad x \in \mathbb{R}^N, \quad \text{and} \quad \lim_{\alpha\to\infty} \varphi_\alpha(|x|) = \varphi_\infty(|x|), \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{1.3}$$

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where $\varphi_\infty = \varphi_\infty(|x|)$ denotes the singular steady state of (1.1) which is given by

$$\varphi_\infty(|x|) := L|x|^{-m}, \quad |x| > 0, \text{ with } m := \frac{2}{p-1} \text{ and } L := \{m(N-2-m)\}^{\frac{1}{p-1}}.$$

Additionally, for $p > p_c$ and $\alpha > 0$ we have

$$\varphi_\alpha(|x|) = L|x|^{-m} - a_\alpha |x|^{-m-\lambda_1} + o(|x|^{-m-\lambda_1}) \quad \text{as } |x| \to \infty, \tag{1.4}$$

where $a_\alpha > 0$ (see [6]) and the constant $\lambda_1 = \lambda_1(N, p)$ is the smaller positive root of

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0. \tag{1.5}$$

It was shown that these steady states are only stable in the case of $p \geq p_c$ (see [6–9]). Concerning the rates of convergence for $p > p_c$, it was shown in [10,11] that if

$$|u_0(x)| \leq \varphi_\infty(|x|), \quad x \in \mathbb{R}^N \setminus \{0\}, \quad \text{and} \quad |u_0(x) - \varphi_\alpha(|x|)| \leq c\,(1+|x|)^{-l}, \quad x \in \mathbb{R}^N, \tag{1.6}$$

hold with some $c > 0$, $\alpha \in \mathbb{R}$ and $l \in (m+\lambda_1, m+\lambda_2+2)$, where λ_2 denotes the larger positive root of (1.5), then

$$\|u(\cdot,t) - \varphi_\alpha(|\cdot|)\|_{L^\infty(\mathbb{R}^N)} \leq C(1+t)^{-\frac{l-m-\lambda_1}{2}}, \quad t \geq 0, \tag{1.7}$$

is satisfied and this estimate is optimal for $\alpha \neq 0$ and cannot be extended to larger values of l.

Concerning the critical case $p = p_c$, Eq. (1.5) now has the double root

$$\lambda := \frac{N-2-2m}{2},$$

and hence $m + \lambda = \frac{N-2}{2}$ is satisfied. Moreover, the regular steady states satisfy

$$\varphi_\alpha(|x|) = L|x|^{-m} - a_\alpha |x|^{-m-\lambda} \ln(|x|) + o(|x|^{-m-\lambda}\ln(|x|)) \quad \text{as } |x| \to \infty, \tag{1.8}$$

for $p = p_c$, where $a_\alpha > 0$ is monotone decreasing in α and depends on N (see [6,12]). As in [13] for the grow-up rate of (1.1) in the case $p = p_c$, the additional logarithmic factor appearing in (1.8) as compared to (1.4) now implies that the convergence rate for $p = p_c$ also differs by a logarithmic factor from the rate given in (1.7) for $p > p_c$. More precisely, we obtain the following result.

Theorem 1.1. *Suppose $N \geq 11$ and $p = p_c$. Moreover, let $u_0 \in C^0(\mathbb{R}^N)$ satisfy (1.6) with some constants $l \in (m+\lambda, m+\lambda+2)$, $\alpha \in \mathbb{R}$, and $c_1 > 0$.*

(a) *There is a positive constant C_1 such that the solution u of (1.1) fulfills*

$$\|u(\cdot,t) - \varphi_\alpha(|\cdot|)\|_{L^\infty(\mathbb{R}^N)} \leq C_1\,(1+t)^{-\frac{l-m-\lambda}{2}}(\ln(t+2))^{-1} \quad \text{for all } t \geq 0. \tag{1.9}$$

(b) *Estimate (1.9) is optimal for any $\alpha \neq 0$ in the following sense. Given $\alpha \neq 0$, there exist initial data u_0 fulfilling (1.6) such that*

$$\|u(\cdot,t) - \varphi_\alpha(|\cdot|)\|_{L^\infty(\mathbb{R}^N)} \geq C_2\,(1+t)^{-\frac{l-m-\lambda}{2}}(\ln(t+2))^{-1} \quad \text{for all } t \geq 0 \tag{1.10}$$

holds with some positive constant C_2.

This work is structured in the following way. In Section 2 we recall some results of [10] concerning certain linearized problems. In Sections 3 and 4 we establish a suitable upper and lower bound for solutions of these linearized problems, respectively, which imply the estimates claimed in Theorem 1.1.

2. The linearized equation

For proving our result, the procedures in [10,11] suggest studying the linearization of (1.1) around its steady states φ_α. Following [10], for $\alpha > 0$ we define the linear operator

$$P_\alpha U := U_{rr} + \frac{N-1}{r}U_r + p\varphi_\alpha^{p-1}U$$

and consider solutions $U = U(r,t)$ of the problem

$$\begin{cases} U_t = P_\alpha U, & r > 0,\ t > 0, \\ U_r(0,t) = 0, & t > 0, \\ U(r,0) = U_0(r), & r \geq 0, \end{cases} \tag{2.1}$$

where U_0 is a continuous function decaying to zero as $r \to \infty$. Furthermore, let $\psi(r)$ satisfy

$$P_\alpha \psi = 0 \quad \text{for } r > 0, \qquad \psi(0) = 1, \qquad \psi_r(0) = 0, \tag{2.2}$$