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On certain subclasses of analytic functions associated with hypergeometric functions

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ABSTRACT

In this paper, we find the necessary and sufficient conditions for functions $zF(a, b; c; z)$ in the generalized class of β uniformly starlike and β uniformly convex functions of order α and also consequences of the results are pointed out.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C}\ |z| < 1\}$. A function $f \in A$ is called *starlike of order* α $(0 \leq \alpha < 1)$ if and only if $\Re(\frac{zf'(z)}{f(z)}) > \alpha \quad (z \in U)$. A function $f \in A$ is called *convex of order* α $(0 \leq \alpha < 1)$ if and only if $\Re(1 + \frac{zf''(z)}{f'(z)}) > \alpha \quad (z \in U)$. We denote the class of all starlike functions of order α by $S^*(\alpha)$ and the class convex functions of order α by $K(\alpha)$. Denote by T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in U, \tag{1.2}$$

$T^*(\alpha)$ and $C(\alpha)$ are the class of starlike and convex functions of order α $(0 \leq \alpha < 1)$, introduced and studied by Silverman [1].

The class β-UCV was introduced by Kanas and Wisniowska [2], where its geometric definition and connections with the conic domains were considered. The class β-UCV was defined pure geometrically as a subclass of univalent functions, that map each circular arc contained in the unit disk U with a center ξ, $|\xi| \leq \beta$ $(0 \leq \beta < 1)$, onto a convex arc. The notion of

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β-uniformly convex function is a natural extension of the classical convexity. Observe that, if $\beta = 0$ then the center ξ is the origin and the class β-UCV reduces to the class of convex univalent functions K. Moreover for $\beta = 1$ corresponds to the class of uniformly convex functions UCV introduced by Goodman [3,4], and studied extensively by Rønning [5,6]. The class β-S_P is related to the class β-UCV by means of the well-known Alexander equivalence between the usual classes of convex K and starlike S^* functions. Further the analytic criterion for functions in these classes is given as below.

For $-1 < \alpha \leq 1$ and $\beta \geq 0$ a function $f \in A$ is said to be in the class

(i) β-uniformly starlike functions of order α is denoted by $S_P(\alpha, \beta)$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in U \tag{1.3}$$

and

(ii) β-uniformly convex functions of order α is denoted by $UCV(\alpha, \beta)$, if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U. \tag{1.4}$$

Indeed it follows from (1.3) and (1.4) that

$$f \in UCV(\alpha, \beta) \Leftrightarrow zf' \in S_P(\alpha, \beta). \tag{1.5}$$

Remark 1.1. It is of interest to state that $UCV(\alpha, 0) = K(\alpha)$ and $S_P(\alpha, 0) = S^*(\alpha)$

Motivated by above definitions we define the following subclasses of A.

For $0 \leq \lambda < 1, 0 \leq \alpha < 1$ and $\beta \geq 0$, we let $S_P(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\mathrm{Re}\left\{\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1\right|, \quad z \in U, \tag{1.6}$$

and also, let $UCV(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\mathrm{Re}\left\{\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} - \alpha\right\} > \beta\left|\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} - 1\right|, \quad z \in U. \tag{1.7}$$

We further let $TS_P(\lambda, \alpha, \beta) = S_P(\lambda, \alpha, \beta) \cap T$ and $UCT(\lambda, \alpha, \beta) = UCV(\lambda, \alpha, \beta) \cap T$. Suitably specializing the parameters we note that

(1) $TS_P(0, \alpha, \beta) = TS_P(\alpha, \beta)$ [7]
(2) $TS_P(0, 0, \beta) = TS_P(\beta)$ [8]
(3) $TS_P(0, \alpha, 1) = TS_P(\alpha)$ [7]
(4) $TS_P(\lambda, \alpha, 0) = T^*(\lambda, \alpha)$ [9]
(5) $TS_P(0, \alpha, 0) = T^*(\alpha)$ [1]
(6) $UCT(0, \alpha, \beta) = UCT(\alpha, \beta)$ [7]
(7) $UCT(0, 0, \beta) = UCT(\beta)$ [10]
(8) $UCT(0, \alpha, 1) = UCT(\alpha)$ [7]
(9) $UCT(\lambda, \alpha, 0) = C(\lambda, \alpha)$ [9]
(10) $UCT(0, \alpha, 0) = C(\alpha)$ [1].

We recall the Gaussian hypergeometric function $F(a, b; c; z)$ defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{(1)_n}, \tag{1.8}$$

where $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \ldots$ and $(a)_n$ is the Pochhammer symbol defined in terms of the Gamma functions, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{Bmatrix} 1 & n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & n \in N \end{Bmatrix}. \tag{1.9}$$

It is known that

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad Re(c-a-b) > 0 \tag{1.10}$$

and the function $F(a, b; c; 1)$ converges if $Re(c - a - b) > 0$.

Carlson and Schaffer [11] studied the class of starlike functions and pre starlike functions involving hypergeometric functions. In 1993, Silverman [12] gave necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $T^*(\alpha)$ and $C(\alpha)$. Motivated by Silverman [12], Swaminathan [13] and Mostafa [14] in this paper, we find the necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $TS_P(\lambda, \alpha, \beta)$ and $UCT(\lambda, \alpha, \beta)$ when $f \in TS_P(\lambda, \alpha, \beta)$ and $f \in UCT(\lambda, \alpha, \beta)$ respectively for a given a, b, c such that $Re(c - a - b) > 0$.