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General fractional f-factor numbers of graphs

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ABSTRACT

Let G be a graph and f an integer-valued function on $V(G)$. Let h be a function that assigns each edge to a number in [0, 1], such that the *f-fractional number* of G is the supremum of $\sum_{e\in E(G)} h(e)$ over all fractional functions h satisfying $\sum_{e\sim v} h(e) \le f(v)$ for every vertex $v \in V(G)$. An *f-fractional factor* is a spanning subgraph such that $\sum_{v\sim e} h(e) = f(v)$ for every vertex v. In this work, we provide a new formula for computing the fractional numbers by using Lovász's Structure Theorem. This formula generalizes the formula given in [Y. Liu, G.Z. Liu, The fractional matching numbers of graphs, Networks 40 (2002) 228–231] for the fractional matching numbers.

1. Introduction

All graphs considered in this work will be simple finite undirected graphs. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We use $d_G(x)$ for the degree of a vertex x in G. For any $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. We write $G - S$ for $G[V(G) - S]$. We refer the reader to [1] for standard graph theoretic terms not defined here.

Let f and g be two nonnegative integer-valued functions on $V(G)$ such that $g(x) \le f(x)$ for every vertex $x \in V(G)$. A spanning subgraph F of G is a *(g, f)-factor* if $g(v) \le d_F(v) \le f(v)$ for all $v \in V(G)$. If $f \equiv g$, then a (g, f)-factor is also called as an *f-factor*. In 1970, Lovász [2] gave a canonical decomposition of $V(G)$ according to its (g, f)-optimal subgraphs. In this work, we only consider $g \equiv f$.

Define $f(S) = \sum_{x\in S} f(x)$. Let $def(G)$ be the *deficiency* of G with respect to an integer-valued function f and be defined as

$$def(G) = \min_{H\subseteq G} \left\{ \sum_{x\in V(H)} |f(x) - d_H(x)| \right\},$$

where H is a spanning subgraph of G. A subgraph H of G is called *f-optimal* if $def(G) = def(H)$. Let M be an f-optimal subgraph; we call $|E(M)|$ the *f-factor number* and denote it by $\mu(G)$. In particular, if $f \equiv 1$, it is usually referred to as the *matching number*. Let $A(G), B(G), C(G), D(G)$ be defined as in Lovász's Structure Theorem (refer to [1]) and for simplicity, denote them by A, B, C, D, respectively. Then

$$C(G) = \{v \in V(G) \mid d_H(v) = f(v) \text{ for every } f\text{-optimal subgraph } H\},$$
$$A(G) = \{v \in V(G) - C(G) \mid d_H(v) \ge f(v) \text{ for every } f\text{-optimal subgraph } H\},$$

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$$B(G) = \{v \in V(G) - C(G) \mid d_H(v) \le f(v) \text{ for every } f\text{-optimal subgraph } H\},$$
$$D(G) = V(G) - A(G) - B(G) - C(G).$$

Let $G[D] = D_1 \cup \cdots \cup D_\tau$. Let M be a subgraph such that $d_M(v) \le f(v)$ for all $v \in A(G)$. For a component D_i of $G[D]$, we refer to D_i as *M-full* if either M contains an edge of $E(D_i, A(G))$ or M misses an edge of $E(V(D_i), B)$; otherwise, D_i is *M-near full*. For any f-optimal subgraph M of G, where $d_M(v) \le f(v)$ for all $v \in A(G)$, the number of nontrivial components of $G[D]$ which are M-near full is denoted by $nc(M)$. Let $nc(G) = \max\{nc(M)\}$, where the maximum is taken over all f-optimal graph M of G with $d_M(v) \le f(v)$ for all $v \in A(G)$. We describe a graph H as *f-critical* if H contains no f-factors, but for any fixed vertex x of $V(H)$, there exists a subgraph K of H such that $d_K(x) = f(x) \pm 1$ and $d_K(y) = f(y)$ for any vertex y ($y \ne x$).

Let h be a function defined on $E(G)$ such that $h(e) \in [0, 1]$ for every $e \in E(G)$ and the *f-fractional number* of G is the supremum of $\sum_{e \in E(G)} h(e)$ over all fractional functions h satisfying $\sum_{e \sim v} h(e) \le f(v)$ for each v. We denote the f-fractional number by $\mu_f(G)$. Let $E_v = \{e \mid e \sim v\}$ and $E^h = \{e \mid e \in E(G) \text{ and } h(e) \ne 0\}$. We call h a fractional *f-indicator function* of G if $h(E_v) = f(v)$ for each $v \in V(G)$. If H is a spanning subgraph of G such that $E(H) = E^h$, then H is called a *fractional f-factor* of G with indicator function h, or simply a fractional f-factor. Let $def_f(G)$ be the deficiency of the fractional f-factor of G, and be defined as

$$def_f(G) = \min\left\{ \sum_{v \in V(G)} \left(\left| \sum_{e \in E_v} h(e) - f(v) \right| \right) \mid h \text{ is a function defined on } E(G) \text{ such that } h(e) \in [0, 1] \text{ for every } e \in E(G) \right\}.$$

In this work, we investigate the relationship between the f-factor number and the fractional f-factor number, provide a new formula for computing the fractional numbers by using Lovász's Structure Theorem, and generalize the formula for the fractional matching number given in [3]. By the definition, clearly

$$\mu_f(G) = \frac{1}{2}(f(V(G)) - def_f(G)).$$

Let f_S be a function on $V(G-S)$ such that $f_S(v) = f(v) - |E(v, S)|$ and f_S^T is the restriction of f_S on the subgraph T of $G-S$. Given an arbitrary f-optimal subgraph F, let $def_F(T)$ denote the deficiency of subset $V(T)$ with respect to the f-factors.

Theorem 1.1 (*Lovász's Structure Theorem*). *Let $D(G)$, $A(G)$, $B(G)$ and $C(G)$ be defined as above. Let F be an f-optimal subgraph. Then*

(i) *every component D_i of $G[D]$ is $f_B^{D_i}$-critical;*
(ii) *for every component D_i of $G[D]$, $def_F(D_i) \le 1$;*
(iii) *$d_F(v) \in \{f(v), f(v)-1, f(v)+1\}$ if $v \in D$; $d_F(v) \le f(v)$ if $v \in B$; $d_F(v) \ge f(v)$ if $v \in A$;*
(iv) *$def(G) = def(F) = f(B) + \tau - f(A) - \sum_{v \in B} d_{G-A}(v)$, where τ denotes the number of components of $G[D]$.*

Lu and Yu [4] gave a different interpretation of $A(G)$, $B(G)$, $C(G)$, $D(G)$ by using alternating trails and thus obtained a shorter proof of Lovász's Structure Theorem. Suppose that F is an f-optimal subgraph, where $d_F(v) \le f(v)$ for all $v \in A(G)$, and let $B_0 = \{v \mid d_F(v) < f(v)\}$. An *$M$-alternating trail* is a trail $P = v_0 v_1 \ldots v_k$ with $v_i v_{i+1} \notin F$ for i even and $v_i v_{i+1} \in F$ for i odd. Then we can define A, B, C, D alternatively as follows:

$$D = \{v \mid \exists \text{ both an even and an odd } F\text{-alternating trail from vertices of } B_0 \text{ to } v\},$$
$$B = \{v \mid \exists \text{ an even } F\text{-alternating trail from a vertex of } B_0 \text{ to } v\} - D,$$
$$A = \{v \mid \exists \text{ an odd } F\text{-alternating trail from a vertex of } B_0 \text{ to } v\} - D,$$
$$C = V(G) - A - B - D.$$

With these new notions, more structural properties of f-optimal subgraphs can be obtained.

Theorem 1.2 (*Lu and Yu [4]*). *Let $D(G)$, $A(G)$, $B(G)$ and $C(G)$ be defined as above. Let F be an arbitrary f-optimal subgraph of G. Then*

(i) *for every component D_i of $G[D]$, if $def(D_i) = 0$, then F either contains an edge of $E(D_i, A)$ or misses an edge of $E(D_i, B)$; if $def(D_i) = 1$, then $E(D_i, B) \subseteq F$ and $E(D_i, A) \cap F = \emptyset$;*
(ii) *if $d_F(v) \le f(v)$ for all $v \in V(G)$, then for any $v \in D$ there are both an even F-alternating trail and an odd F-alternating trail from the vertices of B_0 to v.*

Anstee [5] obtained a formula for the fractional f-factor number.