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Option pricing under some Lévy-like stochastic processes

Rossella Agliardi *

Department Matemates, University of Bologna, viale Filopanti n. 5 - 40126 Bologna, Italy

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ABSTRACT

A generalization of the Lèvy model for financial options is considered which employs pseudodifferential operators with symbols depending on the state variables throughout a small parameter ε. Adapting the classical method of the construction of a parametrix by means of the pseudodifferential calculus an approximate solution to the pricing problem is derived and its implication in terms of the volatility smile, even in very stylized models, is obtained.

1. Introduction

In recent decades the mathematical models based on Lèvy processes – as [1–3] – have replaced the classical option pricing model developed by Black–Scholes and Merton where the underlying asset follows a geometric Brownian motion. The Lèvy class has gained increasing favour in the financial literature and has been employed for pricing also more complex options than the standard ones (see [4–6]). On the other hand a stream of literature has been developed to overcome the inconsistency with market option prices arising from the assumption of constant volatility and to explain such an empirical pattern as the volatility "smile", that is the dependence of the implied Black–Scholes volatilities on the strike of the option under scrutiny. Such variants of the Black–Scholes model separate into two classes of models: the level dependent volatility approach, which describes the underlying asset as a diffusion with volatility depending on the current or past behavior of the asset price, and the stochastic volatility approach, which models the volatility as a further stochastic process, by introducing a new source of randomness. A comparison of the two approaches is contained in [7], where a new model is added to this stream of literature. Empirical work following the theoretical research has generally supported the need for both jumps and stochastic volatility in the underlying asset. [8] merges the two streams of research by incorporating stochastic and mean-reverting volatilities in Lèvy process models, since the otherwise successful Lèvy modeling does not account for the observed variation of option prices across maturity. The stochastic volatility effect is incorporated into the price process by introducing a second process that makes time stochastic and by letting the price process be subordinated by this stochastic "clock". Periods with high volatility are obtained letting time run faster than in periods with low volatility. On the other hand, [9] generalizes the local volatility models—which assume risk neutral dynamics for the stock price of the form: $dS_t = \mu S_t dt + \sigma(S_t, t)dW_t$, being W_t a standard Wiener process—to a Lèvy setting, by defining Lèvy processes which are time changed by an inhomogeneous local speed function, i.e. a deterministic function of time and the level of the process itself. In other words, local volatility is obtained by running the Lèvy process at a speed that depends on the stock price and time. An alternative approach might be taken by introducing state-dependent functions in the generator of the process. This amounts to considering some pseudodifferential operators of order in]0, 2] as generators of Feller processes (see [10])

* Tel.: +39 0512094346.
E-mail address: rossella.agliardi@unibo.it.

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which generalize the classical Lévy case, where the symbols of the generators are independent of the state variable x. In such a framework the price of any contingent claim can be obtained as the solution of a generalized Black–Scholes equation of the form:

$$\partial_t f(t,x) - (r + \psi(x, D_x)) f(t,x) = 0$$

with terminal condition $f(T,x) = g(x)$ which represents the terminal payoff. Here $\psi(x,\xi)$ is assumed to be the characteristic exponent of a Lévy process for each fixed $x \epsilon \mathbb{R}^n$, and the equivalent martingale measure (EMM) requirement $r + \psi(x, -i) = 0$ is supposed to hold. In this paper we follow this approach and build on the classical methods for constructing the fundamental solution of pseudo-differential operators of parabolic type, in order to provide approximate solutions to the pricing problem. This methodology is little explored in the financial literature, [2] being an exception. As an illustration we show that the method is able to provide new formulas that, even in the more traditional case of the Gaussian model, can account for such phenomena as the term structure and the smile of the implicit volatility. This example is discussed in Section 4, while the notation and main definitions are presented in Section 2 and the valuation expression is given in Section 3.

2. Notation

Let us define pseudodifferential operators of class $S^m_{\rho,\delta;\varepsilon,q}(\mathbb{R}^{2n})$ where $m, q \epsilon \mathbb{R}$, $\rho = (\rho_1, \ldots, \rho_n) \epsilon \mathbb{N}^n$, $\delta = (\delta_1, \ldots, \delta_n) \epsilon \mathbb{N}^n$ with $0 \le \delta_j < \rho_j \le 1$, $\varepsilon > 0$. We say that a C^∞-function $\psi(x,\xi)$ defined on $\mathbb{R}^{2n}$ is a symbol of class $S^m_{\rho,\delta;\varepsilon,q}(\mathbb{R}^{2n})$ if for any multi-index α, β there is a constant $C_{\alpha,\beta} \ge 0$ such that:

$$\sup_{x,\xi \epsilon \mathbb{R}^n} |\partial_\xi^\alpha D_x^\beta \psi(x,\xi)| . \varepsilon^{-q-|\beta|} \langle \xi \rangle^{-m+\rho.\alpha-\delta.\beta} \le C_{\alpha,\beta} \tag{2.1}$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ and $D_x = -i\partial_x$. Since these symbols are of the classical type – with only an additional parameter ε – we refer to the textbooks on pseudo-differential operators for the calculus (see [11]). Here a result on the composition of symbols is given for readers' convenience.

Proposition 1. *If* $\psi_j \epsilon S^{m_j}_{\rho,\delta;\varepsilon,q_j}(\mathbb{R}^{2n})$ $(j = 1, 2)$*, then for any N one has:* $(\psi_1 \circ \psi_2)(x,\xi) = \sum_{|\gamma|<N} \frac{1}{\gamma!} \partial_\xi^\gamma \psi_1(x,\xi) D_x^\gamma \psi_2(x,\xi) + r_N(x,\xi)$ *where* $r_N \epsilon S^{m_1+m_2-\theta N}_{\rho,\delta;\varepsilon,q_1+q_2+N}(\mathbb{R}^{2n})$ *with* $\theta = \min_{1\le j\le n}(\rho_j - \delta_j)$.

Proof. Since $r_N(x,\xi) = N \sum_{|\gamma|=N} \int_0^1 \frac{(1-t)^{N-1}}{\gamma!} \left[Os - \iint \mathrm{e}^{-\mathrm{i}y.\eta} \partial_\xi^\gamma \psi_1(x, \xi + t\eta) D_x^\gamma \psi_2(x+y, \xi) \mathrm{d}y \mathrm{đ}\eta \right] \mathrm{d}t$ it is clear that $r_N \epsilon S^{m_1+m_2-\theta N}_{\rho,\delta;\varepsilon,q_1+q_2+N}$ holds. □

In the sequel it will be convenient to consider pseudodifferential operators with the symbols admitting an analytic continuation with respect to ξ into $\mathrm{Im}\,\xi \epsilon \Lambda$ where Λ is an open domain in $\mathbb{R}^n$ whose closure contains the origin. Usually Λ will be of the form $\prod_{j=1}^n]\lambda_j^-, \lambda_j^+[$ with $\lambda_j^- < 0 < \lambda_j^+$. If the symbol ψ admits an analytic continuation w.r.t. ξ into $\mathrm{Im}\,\xi \epsilon \Lambda$ and all its derivatives $\partial_\xi^\alpha D_x^\beta \psi$ admit a continuous extension up to the boundary of Λ and satisfy (2.1), then we will write that $\psi \epsilon S^m_{\rho,\delta;\varepsilon,q}(\mathbb{R}^n \times (\mathbb{R}^n + \mathrm{i}\Lambda))$.

3. An approximate pricing formula

Throughout this paper we consider a model of a financial market with a deterministic saving account e^{rt}, $r \ge 0$, and $n \ge 1$ stocks with the price following a stochastic process $S_t = \mathrm{e}^{X_t}$. In what follows we want to price any contingent claims on the stocks, assuming that it follows a pseudodifferential equation of the form:

$$[\partial_t - r - \psi(x, D_x)] f(t,x) = 0 \tag{3.1}$$

and that the terminal condition $f(T,x) = g(x)$ is given, which represents the terminal payoff of the option. Here $\psi(x, D_x)$ is a pseudodifferential operator whose symbol belongs to $S^m_{\rho,\delta;\varepsilon,0}(\mathbb{R}^n \times (\mathbb{R}^n + \mathrm{i}\Lambda))$ with $m \le 2$, $\Lambda = \prod_{j=1}^n]\lambda_j^-, \lambda_j^+[$, $\lambda_j^- < -1 < 0 < \lambda_j^+$, and satisfies some additional properties that are given below.

A typical case is a Lèvy model, where X_t follows a regular Lévy process of order m and exponential type $[\lambda^-, \lambda^+]$ with characteristic exponent $\psi(\xi)$, i.e. $E(\mathrm{e}^{\mathrm{i}\xi.X_t}) = \mathrm{e}^{-t\psi(\xi)}$, $\forall t \ge 0$. In this case $\psi(\xi) = -i\mu\xi + \phi(\xi)$, where ϕ is holomorphic in the strip $\mathrm{Im}\,\xi \in]\lambda_-, \lambda_+[$, continuous up to the boundary of the strip, $\phi(\xi) \epsilon S^m_{1,0}$ with $m \epsilon]0, 2]$ and there exist $c > 0$ and c_0 such that $\mathrm{Re}\,\phi(\xi) \ge c\langle \xi \rangle^m - c_0$ in $\mathrm{Im}\,\xi \in [\lambda_-, \lambda_+]$. The expectation operator E is taken under an equivalent martingale measure so that $\mathrm{e}^{-rt} S_t$ is a martingale: in terms of the characteristic exponent one has $r + \psi(-i) = 0$.

In this paper a more general framework is considered where the symbol ψ depends also on the state variables x. Then the EMM requirement becomes $r + \psi(x, -i) = 0$ for each fixed $x \epsilon \mathbb{R}^n$. Following [12,11] we will also assume some conditions on $\psi(x,\xi)$ which guarantee the existence of a fundamental solution of the Cauchy problem (3.1) (see Proposition 2) and that are satisfied in many meaningful financial problems.