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Extraconnectivity of hypercubes ☆

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ABSTRACT

Given a graph G and a non-negative integer g, the g-extraconnectivity of G (written $\kappa_g(G)$) is the minimum cardinality of a set of vertices of G, if it exists, whose deletion disconnects G, and where every remaining component has more than g vertices. The usual connectivity and superconnectivity of G correspond to $\kappa_0(G)$ and $\kappa_1(G)$, respectively. In this work, we determine $\kappa_g(Q_n)$ for $0 \le g \le n$, $n \ge 4$, where Q_n denotes the n-dimensional hypercube.

1. Introduction

It is well known that the topology of an interconnected network is often modeled by a connected graph of communication links. In the network the connectivity $\kappa(G)$ is an important factor determining the reliability and fault tolerance of the network. Here, we consider the extraconnectivity, which was defined by Fàbrega and Fiol [2]. The extraconnectivity corresponds to a kind of conditional connectivity introduced by Harary [3].

Let G be a connected undirected graph, and $\mathscr{P}$ a graph-theoretic property. Harary [3] defined the conditional connectivity $\kappa(G;\mathscr{P})$ as the minimum cardinality of a set of vertices, if it exists, whose deletion disconnects G and where every remaining component has property $\mathscr{P}$. Let g be a non-negative integer and let $\mathscr{P}_g$ be the property of having more than g vertices. Fàbrega and Fiol [2] defined the g-*extraconnectivity* $\kappa_g(G)$ of G as $\kappa(G;\mathscr{P})$.

Hypercubes are used as fundamental models for computer networks. There are many research articles on hypercubes (see, for example [4–8]). An n-dimensional hypercube is an undirected graph $Q_n = (V, E)$ with $|V| = 2^n$ and $|E| = n2^{n-1}$. Each vertex can be represented by an n-bit binary string. There is an edge between two vertices whenever their binary string representation differs in only one bit position. It is known that $\kappa_0(Q_n) = \kappa(Q_n) = n$, $\kappa_1(Q_n) = 2n-2$ for $n \ge 3$ and $\kappa_2(G) = 3n-5$ for $n \ge 6$ (see [4,8]). In this work, we show that $\kappa_g(G) = (g+1)n - 2g - \binom{g}{2}$ when $0 \le g \le n-4$ and $\kappa_g(G) = \frac{n(n-1)}{2}$ when $n-3 \le g \le n$ for $n \ge 4$. Following Latifi [5], we express Q_n as $D_0 \odot D_1$, where D_0 and D_1 are the two $(n-1)$-subcubes of Q_n induced by the vertices with the ith coordinates 0 and 1 respectively. Sometimes we use $X^{i-1}0X^{n-i}$ and $X^{i-1}1X^{n-i}$ to denote D_0 and D_1, where $X \in Z_2$. Clearly, the vertex v in one $(n-1)$-subcube has exactly one neighbor v' in another $(n-1)$-subcube; we call v' the *out neighbor* of v. Let $A \subseteq G$, $v \in V(G)$. We use $N_G(v)$ to denote the set of the neighbors of v in G, $N_G(A)$ to denote the set $(\bigcup_{v\in V(A)} N_G(v)) \setminus V(A)$, $C_G(A)$ to denote the set $N_G(A) \cup V(A)$. We follow Bondy [1] for terminologies not given here.

2. Preliminaries

Before discussing the $\kappa_g(Q_n)$, We give the following lemmas.

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Lemma 2.1. *Let A be a subgraph of Q_n with $|V(A)| = g+1$ for $n \geq 4$. Then $|N_{Q_n}(A)| \geq (g+1)n - 2g - \binom{g}{2}$.*

Proof. By induction on $|V(A)|$. Clearly, the result holds for $|V(A)| = 1$. Assume that the result holds for all A with $|V(A)| \leq h$. Next we show that the result is true for A with $|V(A)| = h+1$. We directly use $g+1$ instead of $h+1$.

We first show that Q_n can be decomposed into D_0 and D_1 such that $D_0 \cap A = A_0$ and $D_1 \cap A = A_1$ with $V(A_0) \neq \emptyset$ and $V(A_1) \neq \emptyset$. Since $g \geq 1$, $|V(A)| \geq 2$. Let $x = x_1x_2\cdots x_i\cdots x_n$ and $y = y_1y_2\cdots y_i\cdots y_n$ be two distinct vertices of A; without loss of generality, assume $x_i = 0$ and $y_i = 1$ for an integer i. Let $D_0 = X^{i-1}0X^{n-i}$, $D_1 = X^{i-1}1X^{n-i}$. Then $x \in V(D_0)$, $y \in V(D_1)$, and thus $V(A_0) \neq \emptyset$, $V(A_1) \neq \emptyset$.

Assume $|V(A_0)| = N$, $1 \leq N \leq g$. By the induction hypothesis, we have $|N_{D_0}(A_0)| \geq N(n-1) - 2(N-1) - \binom{N-1}{2}$ and $|N_{D_1}(A_1)| \geq (g+1-N)(n-1) - 2(g-N) - \binom{g-N}{2}$. Since $N_{Q_n}(A) = N_{D_0}(A_0) \cup N_{D_1}(A_0) \cup N_{D_1}(A_1) \cup N_{D_0}(A_1)$ and $N_{D_0}(A_0) \cap N_{D_1}(A_1) = \emptyset$, we have $|N_{Q_n}(A)| \geq |N_{D_0}(A_0)| + |N_{D_1}(A_1)|$. Obviously, $|N_{D_0}(A_0)| + |N_{D_1}(A_1)| - [(g+1)n - 2g - \binom{g}{2}] \geq N(n-1) - 2(N-1) - \binom{N-1}{2} + (g+1-N)(n-1) - 2(g-N) - 2\binom{g-N}{2} - [(g+1)n - 2g - \binom{g}{2}] = -N^2 + (g+1)N - g$. It is easy to see that $f(N) = -N^2 + (g+1)N - g$ is increasing in N when $1 \leq N \leq \frac{g+1}{2}$ and decreasing in N when $\frac{g+1}{2} \leq N \leq g$, and $f(1) = f(g) = 0$. Thus $|N_{D_0}(A_0)| + |N_{D_1}(A_1)| \geq (g+1)n - 2g - \binom{g}{2}$ for $1 \leq N \leq g$. $\square$

Remark 2.2. Note that $h_n(g) = (g+1)n - 2g - \binom{g}{2}$ is increasing when $g \leq n-2$, the maximum of $h_n(g)$ is $h_n(n-2) = (n-1)n - 2(n-2) - \binom{n-2}{2} = h_n(n-1) = n^2 - 2(n-1) - \binom{n-1}{2}$ and $h_n(n-1) = h_n(n-2) > h_n(n) > (g+1)n - 2g - \binom{g}{2}$ for $0 \leq g \leq n-4$. In particular, $h_{n-1}(g_1) + h_{n-1}(g_2) > h_n(g) + 1$ when $0 \leq g_1, g_2 \leq n-1$ and $g_1 + 1 + g_2 + 1 > g + 1$.

Lemma 2.3. *Let $Q_n = D_0 \odot D_1$ and let F be a vertex cutset of Q_n. Suppose B is a subgraph of D_1 consisting of some components of $Q_n - F$. If $|V(B)| \geq g+1$, then $|F| \geq (g+1)n - 2g - \binom{g}{2}$.*

Proof. Let B be the subgraph that satisfies the conditions of this lemma. Assume T_B is a subgraph of B such that $|V(T_B)| = g+1$. By Lemma 2.1, we have $|N_{D_1}(T_B)| \geq (g+1)(n-1) - 2g - \binom{g}{2}$. Since B is disconnected with D_0, that is, for each $v \in N_{D_1}(T_B)$, at least one of v and its out neighbor v' is in F, we thus have $|F| \geq |N_{D_1}(T_B)| + |V(T_B)| \geq (g+1)n - 2g - \binom{g}{2}$. $\square$

Lemma 2.4. *Assume $n \geq 4$, $B \subseteq Q_n$ and $|V(B)| \geq n$. If $|V(Q_n) \setminus C_{Q_n}(B)| \geq n$, then $|N_{Q_n}(B)| > (n-3)n - 2(n-4) - \binom{n-4}{2} + 1$.*

Proof. By induction. Let $B \subseteq Q_n$, $|V(B)| \geq n$ and $|V(Q_n) \setminus C_{Q_n}(B)| \geq n$. For $n = 4$, we have $4 \leq |V(B)| \leq 5$. The result follows directly by Lemma 2.1. Assume that the result holds for all $n < M$. We show that the result is true for $n = M$.

Suppose $|N_{Q_n}(B)| \leq (n-3)n - 2(n-4) - \binom{n-4}{2} + 1$. We shall derive a contradiction. Let $F = N_{Q_n}(B)$, $F_0 = F \cap V(D_0)$ and $F_1 = F \cap V(D_1)$. Then either $|F_0| \leq \frac{(n-3)n-2(n-4)-\binom{n-4}{2}+1}{2}$ or $|F_1| \leq \frac{(n-3)n-2(n-4)-\binom{n-4}{2}+1}{2}$. Without loss of generality, we assume $|F_0| \leq \frac{(n-3)n-2(n-4)-\binom{n-4}{2}+1}{2}$.

Assume that $G_1, G_2, \ldots, G_s$ are all components of $D_0 - F_0$ such that $|V(G_i)| < \frac{n-3}{2}$ and use G^* to denote $D_0 - (F \cup V(G_1 \cup \cdots \cup G_s))$.

Claim 1. $\sum_{i=1}^{s} |V(G_i)| < \frac{n-3}{2}$.

Observe that $(\frac{n-3}{2})n - 2(\frac{n-3}{2} - 1) - \binom{\frac{n-3}{2}-1}{2} > \frac{(n-3)n-2(n-4)-\binom{n-4}{2}+1}{2}$ when n is odd and $(\frac{n-2}{2})n - 2(\frac{n-2}{2} - 1) - \binom{\frac{n-2}{2}-1}{2} > \lfloor\frac{(n-3)n-2(n-4)-\binom{n-4}{2}+1}{2}\rfloor$ when n is even. By Lemma 2.1, $D_0 - F_0$ contains no component A_0 such that $\frac{n-3}{2} \leq |V(A_0)| \leq n$. Now we show that $\sum_{i=1}^{s} |V(G_i)| < \frac{n-3}{2}$.

If $\frac{n-3}{2} \leq \sum_{i=1}^{s} |V(G_i)| \leq n$, by Lemma 2.1, we have $|N_{D_0}(G_1 \cup \cdots \cup G_s)| > |F_0|$, a contradiction.

If $\sum_{i=1}^{s} |V(G_i)| \geq n$, since $|V(G_i)| < \frac{n-3}{2}$, $i = 1, \ldots, s$, we can find a subgraph S consisting of some G_i such that $\frac{n-3}{2} \leq |V(S)| \leq n$. Clearly, $|N_{D_0}(S)| > |F_0|$, a contradiction. Thus $\sum_{i=1}^{s} |V(G_i)| < \frac{n-3}{2}$.

Claim 2. G^* is connected.

Since $|V(D_0) \setminus (F_0 \cup (V(G_1 \cup \cdots \cup G_s)))| > 2^{n-1} - |F_0| - \frac{n-3}{2} > 0$ for $n \geq 4$, thus $V(G^*) \neq \emptyset$.

Suppose G^* is disconnected, then every component of G^* has order at least n. By induction, we have $|F_0 \cup V(G_1 \cup \cdots \cup G_s)| > (n-4)(n-1) - 2(n-5) - \binom{n-5}{2} + 1$. However, $|F_0 \cup V(G_1 \cup \cdots \cup G_s)| < |F_0| + \frac{n-3}{2} \leq \lfloor\frac{(n-3)n-2(n-4)-\binom{n-4}{2}+1}{2}\rfloor + \frac{n-3}{2} \leq (n-4)(n-1) - 2(n-5) - \binom{n-5}{2} + 1$ for $n \geq 5$, a contradiction. Thus G^* is connected.

Assume that $\sum_{i=1}^{s} |V(G_i)| = N$ and $C_1, C_2, \ldots, C_m$ are all components of $D_1 - F_1$ such that $|V(C_i)| \leq n - 3 - N$. Moreover, by Lemma 2.1 and Remark 2.2, we know that $D_1 - F_1$ has no component C_0 such that $n - 2 - N \leq |V(C_0)| \leq n - 1$. Next we derive contradictions by considering two cases.