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Periodic solutions for a degenerate parabolic equation

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ABSTRACT

In this work, we establish the existence of nontrivial nonnegative periodic solutions for a class of degenerate parabolic equations with nonlocal terms. The key is the using of Moser's iteration technique and the theory of the Leray–Schauder degree.

1. Introduction

In recent years, periodic problems for degenerate parabolic equations have been the subject of extensive study; see [1–6] and references therein. In this work, we consider the following periodic degenerate parabolic equation with nonlocal terms:

$$\frac{\partial u}{\partial t} - \Delta u^m = (a - \Phi[u])u, \quad (x, t) \in Q_\omega, \tag{1.1}$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \omega], \tag{1.2}$$

$$u(x, 0) = u(x, \omega), \quad x \in \Omega, \tag{1.3}$$

where $m > 1$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $Q_\omega = \Omega \times (0, \omega)$. This problem models some interesting phenomena in mathematical biology; see [4].

We assume that:

(A1) $\Phi[\cdot] : L^k(\Omega)^+ \to \mathbb{R}^+$ is a bounded and continuous functional satisfying

$$\Phi[u] \le C_1 \|u\|_{L^k(\Omega)}, \quad k > 0,$$

where C_1 is a positive constant independent of u and k, $\mathbb{R}^+ = [0, +\infty)$ and $L^k(\Omega)^+ = \{u \in L^k(\Omega) | u \ge 0, \text{a.e. in } \Omega\}$.

(A2) $a(x, t) \in C_\omega(\overline{Q}_\omega)$ and satisfies

$$\left\{x \in \Omega : \frac{1}{\omega}\int_0^\omega a(x, t)\mathrm{d}t > 0\right\} \ne \emptyset,$$

where $C_\omega(\overline{Q}_\omega)$ denotes the set of functions which are continuous in $\overline{\Omega} \times \mathbb{R}$ and ω-periodic with respect to t.

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It is obvious that (A1), (A2) are weaker than the assumptions in [4], in which the positive continuous functional $\Phi[u]$ is restricted to being bounded by the norm of $\|u\|_{L^2(\Omega)}$ not only from upper but also from lower. The aim of this work is to establish the existence of nontrivial nonnegative periodic solutions for the problem (1.1)–(1.3) under these conditions.

Due to the degeneracy of the equation considered, the problem (1.1)–(1.3) has no classical solutions in general, so we consider generalized solutions in the following sense.

Definition 1.1. A function u is said to be a generalized solution of the problem (1.1)–(1.3), if $u \in L^2(0, \omega; H_0^1(\Omega)) \cap C_\omega(\overline{Q}_\omega)$ and u satisfies

$$\iint_{Q_\omega} \left(-u\frac{\partial \varphi}{\partial t} + \nabla u^m \nabla \varphi - (a - \Phi[u])u\varphi \right) \mathrm{d}x\mathrm{d}t = 0, \tag{1.4}$$

for any $\varphi \in C^1(\overline{Q}_\omega)$ with $\varphi(x, 0) = \varphi(x, \omega)$ and $\varphi|_{\partial\Omega\times(0,\omega)} = 0$.

Our main result is the following theorem.

Theorem 1.1. *If the assumptions* (A1), (A2) *hold, then the problem* (1.1)–(1.3) *admits a nontrivial nonnegative periodic solution.*

2. Preliminaries

Our result will be proved by means of a parabolic generalization. That is we consider the following regularized problem:

$$\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\left((m u_\varepsilon^{m-1} + \varepsilon)\nabla u_\varepsilon\right) = (a - \Phi[u_\varepsilon])u_\varepsilon, \quad (x, t) \in Q_\omega, \tag{2.1}$$

$$u_\varepsilon(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \omega], \tag{2.2}$$

$$u_\varepsilon(x, 0) = u_\varepsilon(x, \omega), \quad x \in \Omega, \tag{2.3}$$

where ε is some positive constant. The desired solution of the problem (1.1)–(1.3) will be obtained as a limit point of the nonnegative solution of the problem (2.1)–(2.3). Now we introduce a map by considering the following problem:

$$\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\left((\tau m u_\varepsilon^{m-1} + \varepsilon)\nabla u_\varepsilon\right) = f, \quad (x, t) \in Q_\omega, \tag{2.4}$$

$$u_\varepsilon(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \omega], \tag{2.5}$$

$$u_\varepsilon(x, 0) = u_\varepsilon(x, \omega), \quad x \in \Omega, \tag{2.6}$$

where $\tau \in [0, 1]$ is a given parameter. For any given $f \in C_\omega(\overline{Q}_\omega)$, $\tau \in [0, 1]$, we define a map $u_\varepsilon = T(\tau, f)$ with $T : [0, 1] \times C_\omega(\overline{Q}_\omega) \to C_\omega(\overline{Q}_\omega)$. With an argument similar to that of [3], we infer that the map $u_\varepsilon = T(\tau, f)$ is well defined and is a compact continuous map. Let $f(v) = (a - \Phi[v])v^+$, where $v^+ = \max\{v, 0\}$. It is easy to see that the existence of the nonnegative solution of the problem (2.1)–(2.3) is equivalent to the existence of the nonnegative fixed point of the map $u_\varepsilon = T(1, f(u_\varepsilon))$.

By standard regularity and the comparison principle we can see that $u_\varepsilon > 0$. Now we will employ Moser's technique to obtain the L^∞ norm bound of u_ε from upper.

Lemma 2.1. *Let u_ε be a nontrivial solution of $u_\varepsilon = T(1, \sigma f(u_\varepsilon))$, $\sigma \in [0, 1]$; then*

$$\|u_\varepsilon(t)\|_{L^\infty(Q_\omega)} < R, \tag{2.7}$$

where $u(t) = u(\cdot, t)$ and R is a positive constant independent of ε and σ.

Proof. Suppose u_ε is a nontrivial solution; then we have $u_\varepsilon > 0$. So u_ε also solves $u_\varepsilon = T(1, \sigma(a - \Phi[u_\varepsilon])u_\varepsilon)$. Multiplying Eq. (2.4) by $u_\varepsilon^{p+1} (p \geq 0)$ and integrating the resulting relation over Ω, noticing that $0 \leq \Phi[u_\varepsilon] \leq \|u_\varepsilon\|_{L^k(\Omega)}$ and $(a - \Phi[u_\varepsilon])u_\varepsilon \leq a u_\varepsilon$, we have

$$\frac{1}{p+2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t)\|_{p+2}^{p+2} + \frac{4m(p+1)}{(p+m+1)^2}\|\nabla u_\varepsilon^{(m+p+1)/2}(t)\|_2^2 \leq M\|u_\varepsilon(t)\|_{p+2}^{p+2},$$

where $M = \sup_{(x,t)\in\overline{Q}_\omega} a(x, t)$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t)\|_{p+2}^{p+2} + C\|\nabla u_\varepsilon^{(m+p+1)/2}(t)\|_2^2 \leq C(p+1)\|u_\varepsilon(t)\|_{p+2}^{p+2}, \tag{2.8}$$

where C denotes various positive constants independent of p and ε. Setting

$$p_k = 2^k + m - 3, \qquad \alpha_k = \frac{2(p_k+2)}{p_k+m+1}, \qquad u_k = u_\varepsilon^{(p_k+m+1)/2}(t), \quad k = 1, 2, \ldots,$$