



PI polynomials of product graphs

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ARTICLE INFO

Article history:

Received 21 August 2007

Received in revised form 13 October 2008

Accepted 6 January 2009

Keywords:

PI index

PI polynomial

Product graph

Hamming graph

Connected graph

ABSTRACT

The Padmakar–Ivan (PI) index of a graph G is defined as $PI(G) = \sum [n_{eu}(e|G) + n_{ev}(e|G)]$, where $n_{eu}(e|G)$ is the number of edges of G lying closer to u than to v , $n_{ev}(e|G)$ is the number of edges of G lying closer to v than to u and summation goes over all edges of G . In this work, the PI polynomial for the Cartesian product graphs is computed.

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1. Introduction

Let G be a graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. As usual, the distance between the vertices u and v of G is denoted by $d(u, v)$ and it is defined as the number of edges in a minimal path connecting the vertices u and v .

A topological index is a real number related to a graph. It must be a structural invariant, i.e., it is fixed by any automorphism of the graph. There are several topological indices have been defined and many of them have found applications as means for modeling chemical, pharmaceutical and other properties of molecules.

The Wiener index W was the first topological index to be used in chemistry. It was introduced in 1947 by Harold Wiener, as the path number for characterization of alkanes [1]. In graph theoretical language, the Wiener index is equal to the count of all shortest distances in a graph. To see a survey on this topic we encourage the reader to consult [2,3].

Let G be a graph and $f = uv$ an edge of G . $n_{fu}(f|G)$ denotes the number of edges lying closer to the vertex u than the vertex v , and $n_{fv}(f|G)$ is the number of edges lying closer to the vertex v than the vertex u . The Padmakar–Ivan (PI) index of a graph G is defined as $PI(G) = \sum_{f \in E(G)} [n_{fu}(f|G) + n_{fv}(f|G)]$; see for details [4–6]. In this definition, edges equidistant from both ends of the edge $f = uv$ are not counted. We call this index the edge PI index and denote it by $PI_e(G)$. Therefore, for every $f = uv \in E(G)$ we define $PI_e(f) = n_{fu}(f|G) + n_{fv}(f|G)$. We also define the vertex PI index of G , $PI_v(G)$, as the sum of $PI_v(f) = [m_{fu}(f|G) + m_{fv}(f|G)]$ over all edges of G , where $m_{fu}(f|G)$ is the number of vertices lying closer to the vertex u than the vertex v and $m_{fv}(f|G)$ is the number of vertices lying closer to the vertex v than the vertex u . Suppose $f \in E(G)$, $N(f) = |E(G)| - PI_e(f)$ and $M(f) = |V(G)| - PI_v(f)$. Then $PI_e(G) = |E(G)|^2 - \sum_{f \in E(G)} N(f)$ and $PI_v(G) = |V(G)||E(G)| - \sum_{f \in E(G)} M(f)$.

Let $e = xy, f = uv \in E(G)$ and $w \in V(G)$. Define $d(w, e) = \min\{d(w, x), d(w, y)\}$. We say that e is parallel to f if $d(x, f) = d(y, f)$. In this case, we write $e \parallel f$. The Cartesian product $G \times H$ of graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \times H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$. If $G_1, G_2, \dots, G_n$ are graphs then we denote $G_1 \times \dots \times G_n$ by $\bigotimes_{i=1}^n G_i$. Suppose $G = \bigotimes_{i=1}^n G_i$. Then for any $k_1, k_2, \dots, k_s \in \{1, 2, \dots, n\}$, we define $\frac{G}{G_{k_1, k_2, \dots, k_s}} = \bigotimes_{i=1, i \neq k_1, k_2, \dots, k_s}^n G_i$.

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In the following lemma, well-known properties of Cartesian product graphs are introduced. We encourage the reader to consult the book of Imrich and Klavzar [7], for more details.

Lemma 1. Let G and H be graphs. Then we have:

- (a) $|V(G \times H)| = |V(G)| \cdot |V(H)|$ and $|E(G \times H)| = |E(G)| \cdot |V(H)| + |V(G)| \cdot |E(H)|$.
- (b) $G \times H$ is connected if and only if G and H are connected.
- (c) If (a, x) and (b, y) are vertices of $G \times H$ then $d_{G \times H}((a, x), (b, y)) = d_G(a, b) + d_H(x, y)$.
- (d) The Cartesian product is commutative and associative. $\square$

In [8], Ashrafi, Manoochehrian and Yousefi-Azari define the PI polynomial of a graph and investigate some of the elementary properties of this polynomial and compute it for some well-known graphs.

Definition. Let G be a connected graph and u, v be vertices of G . We define

$$N(u, v) = \begin{cases} n_{fu}(f|G) + n_{fv}(f|G) & f = uv \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

Then the PI polynomial of G is defined as $PI(G; x) = \sum_{\{u,v\} \subseteq V(G)} x^{N(u,v)}$ and we have

$$\begin{aligned} PI(G; x) &= \sum_{\{u,v\} \subseteq V(G)} x^{N(u,v)} = \sum_{(u,v) \in E(G)} x^{N(u,v)} + \sum_{(u,v) \notin E(G)} 1 \\ &= \sum_{f \in E(G)} x^{Pl_e(f)} + \binom{|V(G)| + 1}{2} - |E(G)| \\ &= \sum_{f \in E(G)} x^{|E(G)| - N(f)} + \binom{|V(G)| + 1}{2} - |E(G)|. \end{aligned}$$

The Wiener index of Cartesian product graphs was studied in [9,10]. In [11], Khalifeh, Yousefi-Azari and Ashrafi computed the PI index of Cartesian product graphs. Here we continue this progress by computing the PI polynomial of Cartesian product graphs. We prove that:

Theorem 2. Let $G_1, G_2, \dots, G_n$ be connected graphs and $G = \bigotimes_{i=1}^n G_i$; then we have

$$PI\left(\bigotimes_{i=1}^n G_i; x\right) = \sum_{i=1}^n \left| V\left(\frac{G}{G_i}\right) \right| \left[\sum_{f \in E(G_i)} x^{|E(\frac{G}{G_i})| Pl_v(f) + |V(\frac{G}{G_i})| Pl_e(f)} \right] + \varphi(G)$$

where $\varphi(G) = \binom{|V(G)|+1}{2} - |E(G)|$.

Corollary. If G is a connected graph and n is a positive integer then

$$PI(G^n; x) = n|V(G)|^{n-1} \left[\sum_{f \in E(G)} x^{(n-1)|E(G)| + |V(G)|^{n-2} Pl_v(f) + |V(G)|^{n-1} Pl_e(f)} \right] + \varphi(G).$$

Throughout this work, we only consider connected graphs and let K_n denote the complete graph. Our notation is standard and taken mainly from [12–22].

2. Proof of theorem

In this section, we prove the main result of this work.

Lemma 2. Suppose G_1 and G_2 are arbitrary graphs and $e = (x, s)(y, s), f = (t, a)(t, b) \in E(G_1 \times G_2)$. Then the following properties are equivalent:

- (i) $e \parallel f$.
- (ii) $d_{G_1}(x, t) = d_{G_1}(y, t)$.
- (iii) $d_{G_2}(a, s) = d_{G_2}(b, s)$.

Proof. (i) $\Leftrightarrow$ (ii)

Let $G = G_1 \times G_2$ and $xy \in E(G_1), ab \in E(G_2)$. Then $d_{G_1}(x, t) = d_{G_1}(y, t)$ if and only if $(x, s)(y, s) \parallel (t, a)(t, b)$. Consider the following equalities:

$$\begin{aligned} d_G((x, s), (t, a)(t, b)) &= d_G((y, s), (t, a)(t, b)) \\ \text{Min}\{d_G((x, s), (t, a)), d_G((x, s), (t, b))\} &= \text{Min}\{d_G((y, s), (t, a)), d_G((y, s), (t, b))\} \\ d_{G_1}(x, t) + \text{Min}\{d_{G_2}(s, a), d_{G_2}(s, b)\} &= d_{G_1}(y, t) + \text{Min}\{d_{G_2}(s, a), d_{G_2}(s, b)\} \\ d_{G_1}(x, t) &= d_{G_1}(y, t). \end{aligned}$$

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