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The exponent of Cartesian product of cycles

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ABSTRACT

A digraph D is primitive if for each pair of vertices v, w of D, there is a positive integer k such that there is a directed walk of length k from v to w. The minimum of such k is the exponent of D. In this paper, we show that for a primitive graph G and a strongly connected bipartite digraph D, the exponent of the Cartesian product $G \times D$ is equal to the addition of the exponent of G and the diameter of D. Finally, we find the exponents of Cartesian products of cycles.

1. Introduction

Let $D = (V, A)$ be a digraph. For each pair v, w of vertices of D, a directed $v \longrightarrow w$ walk of length p is a sequence of vertices $v = v_0, v_1, \ldots, v_p = w$ and a sequence of arcs $(v_0, v_1), \ldots, (v_{p-1}, v_p)$ in D. For each pair v, w of vertices in D we use $v \stackrel{\alpha}{\longrightarrow} w$ if there is a directed $v \longrightarrow w$ walk of length α. Conventionally, $v \stackrel{0}{\longrightarrow} v$ is permitted. A digraph D is primitive if there is a positive integer k such that for any given pair of vertices v, w in D, $v \stackrel{k}{\longrightarrow} w$. We say that the smallest such value k is the exponent, $\exp(D)$, of D. After the classical work of Wielandt [1], many results have been obtained [2–10] on the exponents of graphs and digraphs.

Definition 1. The Cartesian product $D_1 \times D_2$ of the digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ is a digraph $(V_1 \times V_2, A)$ such that

$$A = \{((v_1, w_1), (v_2, w_2)) \in (V_1 \times V_2) \times (V_1 \times V_2) | v_1 = v_2 \text{ and } (w_1, w_2) \in A_2, \text{ or } w_1 = w_2 \text{ and } (v_1, v_2) \in A_1\}.$$

The Cartesian product of graphs has been used in the theory of interconnection networks [11,12].

Definition 2. For each pair v, w of vertices in a digraph $D = (V, A)$, the distance $\mathrm{dist}(v, w)$ from v to w is the smallest α such that $v \stackrel{\alpha}{\longrightarrow} w$ and if there is no directed walk from v to w, $\mathrm{dist}(v, w) = \infty$. The diameter of D is defined by

$$\mathrm{diam}(D) \equiv \sup_{v,w \in V} \mathrm{dist}(v, w).$$

For a strongly connected digraph D, $\mathrm{diam}(D) < \infty$.

Lamprey and Barnes [13] showed that

$$\exp(D \times E) \le (n + m)^2 - 4(n + m) + 5$$

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for a digraphs D and E on n and m vertices respectively. Kim et al. [14] improved the upper bound to $nm - 1$, which is extremal when $(n, m) = 1$. In this paper, we show that for a primitive graph G and a strongly connected bipartite digraph D,

$$\exp(G \times D) = \exp(G) + \mathrm{diam}(D).$$

By using the above formula, we compute the exponent of the Cartesian product of cycles.

2. Main theorems

Lemma 1. *Let D, E be digraphs, v_1, v_2 be vertices in D and w_1, w_2 be vertices in E. If $(v_1, w_1) \stackrel{\alpha}{\longrightarrow} (v_2, w_2)$ in $D \times E$, then $v_1 \stackrel{\beta}{\longrightarrow} v_2$ and $w_1 \stackrel{\gamma}{\longrightarrow} w_2$ for some β and γ such that $\alpha = \beta + \gamma$.*

Proof. Let $W : (v_1, w_1) = (x_0, y_0) \longrightarrow (x_1, y_1) \longrightarrow \cdots \longrightarrow (x_\alpha, y_\alpha) = (v_2, w_2)$ be a walk in $D \times E$. Let $S_1 = \{i | x_{i-1} = x_i\}$ and $S_2 = \{i | y_{i-1} = y_i\}$. Then, if $1 \le i \le \alpha$, since $((x_{i-1}, y_{i-1}), (x_i, y_i))$ is an arc of $D \times E$, $x_{i-1} = x_i$ and (y_{i-1}, y_i) is an arc of E or $y_{i-1} = y_i$ and (x_{i-1}, x_i) is an arc of D. Thus $S_1 \cup S_2 = \{1, 2, \ldots, \alpha\}$. If $i \in S_1 \cap S_2$, $x_{i-1} = x_i$ and $y_{i-1} = y_i$. So there is a loop at x_i in D or a loop at y_i in E. Let $S_3 = \{i \in S_1 \cap S_2 |$ there is a loop at x_i in $D\}$ and $S_4 = (S_1 \cap S_2) \setminus S_3$. Let $T_1 = S_1 \setminus S_3$ and $T_2 = S_2 \setminus S_4$. Then, $T_1 \cup T_2 = \{1, 2, \ldots, \alpha\}$ and $T_1 \cap T_2 = \phi$. Note that if $i \in T_1$, (y_{i-1}, y_i) is an arc of E and if $i \in T_2$, (x_{i-1}, x_i) is an arc of D. Let $T_1 = \{i_1 < i_2 < \cdots < i_\gamma\}$ and $T_2 = \{j_1 < j_2 < \cdots < j_\beta\}$. Then, $\beta + \gamma = \alpha$. If $1 \le i < i_1$, since $i \in T_2$, $y_{i-1} = y_i$. So $w_1 = y_0 = y_1 = \cdots = y_{i_1 - 1}$. Thus (y_0, y_{i_1}) is an arc of E. If $i_s < i < i_{s+1}$, since $i \in T_2$, $y_{i-1} = y_i$. So $y_{i_s} = y_{i_s+1} = \cdots = y_{i_{s+1}-1}$. Thus $(y_{i_s}, y_{i_{s+1}})$ is an arc of E. If $i_\gamma < i \le \alpha$, since $i \in T_2$, $y_{i-1} = y_i$. So $y_{i_\gamma} = y_{i_\gamma+1} = \cdots = y_\alpha = w_2$. Therefore $w_1 = y_0 \longrightarrow y_{i_1} \longrightarrow y_{i_2} \longrightarrow \cdots \longrightarrow y_{i_\gamma} = y_\alpha = w_2$. This implies $w_1 \stackrel{\gamma}{\longrightarrow} w_2$. Similarly, $v_1 \stackrel{\beta}{\longrightarrow} v_2$. □

Corollary 1 (*K. Day and A. Al-Ayyoub [11]*). *Let D, E be strongly connected digraphs. Then,*

$$\mathrm{diam}(D \times E) = \mathrm{diam}(D) + \mathrm{diam}(E).$$

Proof. Let (v_1, w_1) and (v_2, w_2) be in $D \times E$ such that $\mathrm{dist}((v_1, w_1), (v_2, w_2)) = \mathrm{diam}(D \times E)$. Let $v_1 \stackrel{\alpha}{\longrightarrow} v_2$, $w_1 \stackrel{\beta}{\longrightarrow} w_2$ with $\alpha \le \mathrm{diam}(D)$, $\beta \le \mathrm{diam}(E)$. Then $(v_1, w_1) \stackrel{\alpha}{\longrightarrow} (v_2, w_1) \stackrel{\beta}{\longrightarrow} (v_2, w_2)$. Thus, $(v_1, w_1) \stackrel{\alpha+\beta}{\longrightarrow} (v_2, w_2)$. Therefore,

$$\mathrm{diam}(D \times E) = \mathrm{dist}((v_1, w_1), (v_2, w_2)) \le \alpha + \beta \le \mathrm{diam}(D) + \mathrm{diam}(E).$$

Conversely, let $\alpha = \mathrm{diam}(D)$ and $\beta = \mathrm{diam}(E)$. Then there exist v_1, v_2, w_1 and w_2 such that $\mathrm{dist}(v_1, v_2) = \alpha$ and $\mathrm{dist}(w_1, w_2) = \beta$. If $(v_1, w_1) \stackrel{\gamma}{\longrightarrow} (v_2, w_2)$, then by Lemma 1 $v_1 \stackrel{\alpha'}{\longrightarrow} v_2$ and $w_1 \stackrel{\beta'}{\longrightarrow} w_2$ such that $\gamma = \alpha' + \beta'$. But, $\alpha' \ge \alpha$ and $\beta' \ge \beta$. Thus, $\gamma = \alpha' + \beta' \ge \alpha + \beta = \mathrm{diam}(D) + \mathrm{diam}(E)$. Therefore,

$$\mathrm{diam}(D \times E) \ge \mathrm{diam}(D) + \mathrm{diam}(E). \quad \square$$

Theorem 1. *If G is a primitive graph and D is a strongly connected bipartite digraph, then*

$$\exp(G \times D) = \exp(G) + \mathrm{diam}(D).$$

Proof. Let $G = (V, E)$, $D = (W, A)$, $\alpha = \exp(G)$ and $\beta = \mathrm{diam}(D)$. Let $v, v' \in V$ and $w, w' \in W$. Then, $w \stackrel{t}{\longrightarrow} w'$ for some $t \le \beta$. Since $\alpha + \beta - t \ge \alpha$, $v \stackrel{\alpha+\beta-t}{\longrightarrow} v'$. Since $(v, w) \stackrel{\alpha+\beta-t}{\longrightarrow} (v', w) \stackrel{t}{\longrightarrow} (v', w')$, $(v, w) \stackrel{\alpha+\beta}{\longrightarrow} (v', w')$.
Conversely, there are $v_1, v_2 \in V$ and $w_1, w_2 \in W$ such that there is no directed walk of length $\alpha - 1$ from v_1 to v_2 and $\mathrm{dist}(w_1, w_2) = \beta$. If $(v_1, w_1) \stackrel{\alpha+\beta-1}{\longrightarrow} (v_2, w_2)$, from Lemma 1, $v_1 \stackrel{\gamma}{\longrightarrow} v_2$ and $w_1 \stackrel{\delta}{\longrightarrow} w_2$ for some γ and δ such that $\alpha + \beta - 1 = \gamma + \delta$. Since $\mathrm{dist}(w_1, w_2) = \beta$ and D is bipartite, $\delta - \beta = 2t$ for some non-negative integer t. Thus $\alpha - 1 = \gamma + 2t$. Since G is a graph, $v_2 \stackrel{2t}{\longrightarrow} v_2$, which implies $v_1 \stackrel{\alpha-1}{\longrightarrow} v_2$ in G. We obtain a contradiction. Therefore $\exp(G \times D) = \exp(G) + \mathrm{diam}(D)$. □

Since the path P_n and the directed cycle Z_{2k} are bipartite, we have the following Corollaries 2 and 3.

Corollary 2. *If G is a primitive graph and P_n is a path on n vertices, then*

$$\exp(G \times P_n) = \exp(G) + n - 1.$$

Corollary 3. *If G is a primitive graph and Z_{2k} is a directed cycle on $2k$ vertices, then*

$$\exp(G \times Z_{2k}) = \exp(G) + 2k - 1.$$