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Some common fixed point theorems for a pair of tangential mappings in symmetric spaces

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ABSTRACT

A metrical common fixed point theorem for a pair of self mappings due to Sastry and Murthy (K.P.R. Sastry, I.S.R. Krishna Murthy, A common fixed points of two partially commuting tangential selfmaps on a metric space, J. Math. Anal. Appl. 250 (2000) 731734.) [8] is extended to symmetric spaces which in turn generalises a fixed point theorem due to Pant (R.P. Pant, Common fixed points of Lipschitz type mapping pairs, J. Math. Anal. Appl. 248 (1999) 280283.) [11] besides deriving some related results. Some illustrative examples to highlight the realised improvements are also furnished.

1. Introduction with preliminaries

A symmetric d (introduced by K. Menger in 1928) on a non-empty set X is a function $d : X \times X \to [0, \infty)$ which satisfies $d(x, y) = d(y, x)$ and $d(x, y) = 0 \Leftrightarrow x = y$ (for all $x, y \in X$). If d is a symmetric on a set X, then for $x \in X$ and $\epsilon > 0$, we write $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. A topology $\tau(d)$ on X is given by the sets U (along with empty set) in which for each $x \in U$, $B(x, \epsilon) \subset U$ for some $\epsilon > 0$. A set $S \subset X$ is a neighbourhood of $x \in X$ if and only if there is a U containing x such that $x \in U \subset S$. A symmetric d is said to be a semi-metric if for each $x \in X$ and for each $\epsilon > 0$, $B(x, \epsilon)$ is a neighbourhood of x in the topology $\tau(d)$. Thus a symmetric (resp. a semi-metric) space X is a topological space whose topology $\tau(d)$ on X is induced by a symmetric (resp. a semi-metric d). Notice that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $\tau(d)$. The distinction between a symmetric and a semi-metric is apparent as one can easily construct a semi-metric d such that $B(x, \epsilon)$ need not be a neighbourhood of x in $\tau(d)$.

Since a symmetric space is not essentially Hausdorff, therefore in order to prove fixed point theorems some additional axioms are required. The following axioms are relevant to this note which are available in Galvin and Shore [1], Wilson [2] and Aliouche [3]. From now on symmetric as well as semi-metric spaces will be denoted by (X, d).

Definition 1.1 ((W_3): *(cf. [2])*). Given $\{x_n\}$, x and y in X, $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$ imply $x = y$.

Definition 1.2 ((W_4): *(cf. [2])*). Given $\{x_n\}$, $\{y_n\}$ and x in X, $d(x_n, x) \to 0$ and $d(x_n, y_n) \to 0$ imply $d(y_n, x) \to 0$.

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Definition 1.3 ((*HE*): *(cf. [3])*). Given $\{x_n\}$, $\{y_n\}$ and x in X, $d(x_n, x) \to 0$ and $d(y_n, x) \to 0$ imply $d(x_n, y_n) \to 0$.

Definition 1.4 ((1*C*): *(cf. [1])*). A symmetric d is said to be 1-continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$.

Definition 1.5 ((*CC*): *(cf. [1])*). A symmetric d is said to be continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$ where $\{x_n\}$, $\{y_n\}$ are sequences in X and $x, y \in X$.

Clearly, the continuity of a symmetric is a stronger property than 1-continuity (i.e. (*CC*) implies (1*C*) but not conversely). Also (W_4) implies (W_3) whereas (1*C*) implies (W_3) but the converse implications are not true in general. All other possible implications amongst (W_3), (W_4), (*HE*) and (1*C*) are not generally true.

As usual, a sequence $\{x_n\}$ in a semi-metric space (X, d) is said to be d-Cauchy sequence if it satisfies the standard metric condition. It is interesting to note that in a semi-metric space, Cauchy convergence criterion is not a necessary condition for the convergence of a sequence but this criterion becomes a necessary condition if semi-metric is suitably restricted (see Wilson [2]). In [4], Burke furnished an illustrative example to show that a convergent sequence in a semi-metric space need not admit a Cauchy subsequence. But he was able to formulate an equivalent condition under which every convergent sequence in a semi-metric space admits a Cauchy subsequence. There are several concept of completeness in semi-metric spaces e.g. S-completeness, d-Cauchy completeness, strong and weak completeness whose details are available in Wilson [2], but we omit the details as such notions are not relevant to this note.

Finally, we list the remaining relevant definitions to our presentation.

Definition 1.6. We recall that a pair of self-mappings (f, g) defined on a symmetric (or semi-metric) space (X, d) is said to be

(i) compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X,
(ii) R-weakly commuting (cf. [5]) on X if $d(fgx, gfx) \le Rd(fx, gx)$ for some $R > 0$ where x varies over X,
(iii) pointwise R-weakly commuting (cf. [5]) on X if given x in X there exists $R > 0$ such that $d(fgx, gfx) \le Rd(fx, gx)$,
(iv) non-compatible (cf. [6]) if there exists some sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X but $\lim_{n\to\infty}(fgx_n, gfx_n)$ is either non-zero or non-existent,
(v) tangential (or satisfying property (E.A)) (cf. [7,8]) if there exists a sequence $\{x_n\}$ in X and some $t \in X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$,
(vi) partially commuting (or weakly compatible or coincidentally commuting) (cf. [9]) if pair commutes on the set of coincidence points and
(vii) occasionally weakly compatible (in short OWC) (cf. [10]) if there is at least one coincidence point x of (f, g) in X at which (f, g) commutes.

Definition 1.7. let f and g be two self maps defined on a symmetric space (X, d). Then f is said to be g-continuous (cf. [8]) if $gx_n \to gx \Rightarrow fx_n \to fx$ whenever $\{x_n\}$ is a sequence in X and $x \in X$.

Notice that pointwise R-weak commutativity is equivalent to commutativity at coincidence points whereas compatible maps are pointwise R-weakly commutating as they commute at their coincidence points. Interestingly, the class of tangential maps contains as proper subsets the classes of compatible as well as non-compatible maps and this is the motivation to use the tangential property (or property (E.A)) instead of compatibility or non-compatibility.

For the sake of completeness, we state the following theorem contained in Pant [11].

Theorem 1.1. *Let (f, g) be a pair of non-compatible pointwise R-weakly commuting self-mappings of a metric space (X, d) satisfying*

(a_1) $\overline{f(X)} \subset g(X)$,
(a_2) $d(fx, fy) \le kd(gx, gy)$, *for all* $x, y \in X$, $k \ge 0$, *and*
(a_3) $d(fx, f^2x) \ne \max\{d(fx, gfx), d(f^2x, gfx)\}$ *whenever the right-hand side is non-zero. Then*
(c_1) *f and g have a common fixed point.*

As an extension of Theorem 1.1, Sastry and Murthy [8] proved the following result:

Theorem 1.2. *If (in the setting of Theorem 1.1)* (a_3) *holds and further*

(a_4) *the pair (f, g) is partially commuting,*
(a_5) *the pair (f, g) is tangential,*
(a_6) *f is g-continuous, and*
(a_7) *either $\overline{f(X)} \subset g(X)$ or $g(X)$ is closed, then*
(c_2) *f and g have a common fixed point.*