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Zero-divisor graphs of partially ordered sets[☆]

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ABSTRACT

Let $(P, \leq)$ be a partially ordered set (poset, briefly) with a least element 0 and $S \subseteq P$. An element $x \in P$ is a lower bound of S if $s \geq x$ for all $s \in S$. A simple graph $G(P)$ is associated to each poset P with 0. The vertices of the graph are labeled by the elements of P, and two vertices x, y are connected by an edge in case 0 is the only lower bound of $\{x, y\}$ in P. We show that if the chromatic number $\chi(G(P))$ and the clique number $\omega(G(P))$ are finite, then $\chi(G(P)) = \omega(G(P)) = n + 1$ in which n is the number of minimal prime ideals of P.

1. Introduction

In [1], Beck associated to any commutative ring R a simple graph $G(R)$ whose vertices are labeled by the elements of R, and with two vertices adjacent (connected by an edge) if $x \cdot y = 0$. The problem Beck studied was the chromatic number $\chi(G(R))$ and the clique number $\omega(G(R))$ of $G(R)$. Beck conjectured that $\chi(G(R)) = \omega(G(R))$ for an arbitrary ring R, but Beck's question was settled in the negative in [2].

Recently in [3], DeMeyer et al. defined the zero-divisor graph of a commutative semigroup S with zero ($0x = 0$ for all $x \in S$). The zero-divisor graph of a commutative semigroup with zero is a graph whose vertices are the nonzero zero divisors of the semigroup, with two distinct vertices joined by an edge whenever their product is zero. They have shown that the number of minimal ideals of S gives a lower bound to the clique number of S. The zero-divisor graph of various algebraic structures has been studied by several authors [4–8].

In this paper, to any poset $(P, \leq)$ with a least element 0 we first define the zero-divisor graph, denoted by $G(P)$, of P. We show that if the chromatic number $\chi(G(P))$ and the clique number $\omega(G(P))$ are both finite, then $\chi(G(P)) = \omega(G(P)) = n+1$ in which n is the number of minimal prime ideals of P.

2. Definition and notation of graphs and partially ordered sets

Next we introduce some definitions and notations on graphs and partially ordered sets. We use the standard terminology of graphs in [9] and partially ordered sets in [10].

Recall that the complete graph K_n of order n is a simple graph with n vertices in which every vertex is adjacent to every other vertex. A *clique* in a graph is a set of pairwise adjacent vertices. Since any subgraph induced by a clique is a complete subgraph, the two terms and their notations are usually used interchangeably. A k-clique is a clique of order k. The *clique*

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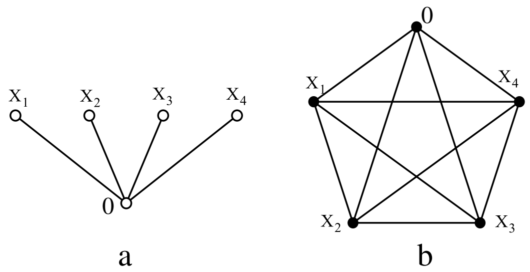

Fig. 1. (a) Hasse diagram of P; (b) The associated zero-divisor graph $G(P)$ of P.

number $\omega(G)$ of a graph G is the order of a largest clique in G. Graph coloring is a special case of graph labeling. It is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. Vertex coloring is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color. The smallest number of colors needed to color a graph G is called its *chromatic number* $\chi(G)$. Since vertices of a clique require distinct colors, we have the following result.

Lemma 1. *For every graph G, $\chi(G) \geq \omega(G)$.*

Given a partially ordered set $(P, \leq)$ (poset, briefly) with a least element 0 and $S \subseteq P$. An element $x \in P$ is a lower bound of S if $s \geq x$ for all $s \in S$. An upper bound is defined dually. The set of all lower bounds of S is denoted by S^ℓ and the set of all upper bounds by S^u, where $S^\ell := \{x \in P \mid (\forall s \in S)\ s \geq x\}$ and $S^u := \{x \in P \mid (\forall s \in S)\ s \leq x\}$. Let $A^{\ell u} = (A^\ell)^u$. We present some properties of poset in the following. They are necessary to show our main result in the next section.

Lemma 2. *For a poset $(P, \leq)$, if A and B are subsets of P, then*

(1) *$A \subseteq A^{\ell u}$ and $A \subseteq A^{u\ell}$;*
(2) *If $A \subseteq B$, then $A^\ell \supseteq B^\ell$ and $A^u \supseteq B^u$;*
(3) *$A^\ell = A^{\ell u \ell}$ and $A^u = A^{u\ell u}$.*

Proof. For $\forall a \in A$ and $\forall x \in A^\ell$ we have $a \geq x$, which says that $A \subseteq (A^\ell)^u = A^{\ell u}$. Dually, $A \subseteq A^{u\ell}$. Thus (1) holds.

If $A \subseteq B$, then any element of B^ℓ is a lower bound of B and so is a lower bound of A and hence belongs to A^ℓ. Thus, obviously (2) holds.

By (1) we have $A \subseteq A^{\ell u}$, further by (2), $A^\ell \supseteq (A^{\ell u})^\ell = A^{\ell u \ell}$. But (1) applied to A^ℓ gives $A^\ell \subseteq (A^\ell)^{u\ell} = A^{\ell u \ell}$. Hence $A^\ell = A^{\ell u \ell}$ and by duality $A^u = A^{u\ell u}$. □

For a nonempty subset I of P, I is an *order-ideal* of P if, $\forall x, y \in P$, $x \in I$, $y \leq x$, we have $y \in I$. A proper order-ideal I of P is called *prime* if $\forall x, y \in P$, $\{x, y\}^\ell \subseteq I$ then $x \in I$ or $y \in I$. For a poset $(P, \leq)$ and a proper ideal I of P, I is called *n-prime* $(n \geq 2)$ if for any $x_1, \ldots, x_n \in P$, $\{x_1, \ldots, x_n\}^\ell \subseteq I$ implies $x_i \in I$ for some $i \in \{1, \ldots, n\}$.

Lemma 3. *For every poset $(P, \leq)$, if I is a proper ideal of P, then for all $n \geq 2$, I is prime if and only if I is n-prime.*

Proof. We first show that any prime ideal I of P is n-prime for all $n \geq 2$. We proceed by induction on n. Clearly, I is 2-prime if and only if I is prime. So for $n = 2$ the result holds.

Assume the statement holds for $n - 1$. Let $\{x_1, \ldots, x_{n-1}, x_n\}^\ell \subseteq I$ for some prime ideal I of P. Take $y \in \{x_1, \ldots, x_{n-1}\}^\ell$ arbitrarily. Then $\{y, x_n\}^\ell \subseteq \{x_1, \ldots, x_{n-1}, x_n\}^\ell \subseteq I$. If $x_n \notin I$, then by the primeness of I, $y \in I$. Since y is an arbitrarily element of $\{x_1, \ldots, x_{n-1}\}^\ell$ and $x_n \notin I$, we have $\{x_1, \ldots, x_{n-1}\}^\ell \subseteq I$. By induction hypothesis we conclude that $x_i \in I$ for some $i \in \{1, \ldots, n-1\}$. The converse holds trivially. This complete the proof. □

To any poset $(P, \leq)$ with a least element 0 we define the *zero-divisor graph*, denoted by $G(P)$, of P as follows: its vertices are just the elements of P, and $x, y \in G(P)$ are connected by an edge if and only if $\{x, y\}^\ell = \{0\}$. The *chromatic number* $\chi(P)$ of P is the chromatic number of $G(P)$. A subset C of P is called a *clique* if $\forall x, y \in C$, $\{x, y\}^\ell = \{0\}$.

Recall the symbol $\parallel$ used to denote non-comparability. Let $(P, \leq)$ be a poset in which the element set of $P = \{0, x_1, \ldots, x_n\}$, $0 \leq x_i$ for $i \in \{1, \ldots, n\}$ and $x_i \parallel x_j$ for $i \neq j$. For $n = 4$, the Hasse diagram of P and the associated zero-divisor graph $G(P)$ of P see Fig. 1(a) and (b), respectively.

Let $(P, \leq)$ be a poset. P is called a *chain* if, $\forall x, y \in P$, either $x \leq y$ or $y \leq x$ (that is, if any two elements of P are comparable). The poset P is an *antichain* if $x \leq y$ in P only if $x = y$. Let $(P, \leq)$ be a chain in which the element set of $P = \{0, x_1, \ldots, x_n\}$, $0 \leq x_i$ for $i \in \{1, \ldots, n\}$ and $x_i \leq x_{i+1}$. For $n = 4$, the Hasse diagram of P and the associated zero-divisor graph $G(P)$ of P see Fig. 2(a) and (b), respectively.

Let $(P, \leq)$ be a poset with the element set $P = \{0, x_1, \ldots, x_n, y_1, \ldots, y_m\}$, in which $0 \leq x_i$, $0 \leq y_j$, $x_i \parallel y_j$ and $x_i \leq x_{i+1}$, $y_j \leq y_{j+1}$ for $1 \leq i \leq n$, $1 \leq j \leq m$. For $n = 4$ and $m = 3$, the Hasse diagram of P and the associated zero-divisor graph $G(P)$ of P see Fig. 3(a) and (b), respectively.