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Minimally restricted edge connected graphs☆

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Abstract

For a connected graph $G = (V, E)$, an edge set $S \subset E$ is a restricted edge cut if $G - S$ is disconnected and there is no isolated vertex in $G - S$. The cardinality of a minimum restricted edge cut of G is the restricted edge connectivity of G, denoted by $\lambda'(G)$. A graph G is called minimally restricted edge connected if $\lambda'(G - e) < \lambda'(G)$ for each edge $e \in E$. A graph G is λ'-optimal if $\lambda'(G) = \xi(G)$, where $\xi(G)$ is the minimum edge degree of G. We show in this work that a minimally restricted edge connected graph is always λ'-optimal.

Keywords: Restricted edge connectivity

1. Introduction

A network can be conveniently modelled as a graph $G = (V, E)$. A classic measure of the fault tolerance of a network is the edge connectivity $\lambda(G)$. In general, the larger $\lambda(G)$ is, the more reliable the network is [3]. For $\lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G, a graph is called λ-*optimal* if $\lambda(G) = \delta(G)$. There are many sufficient conditions for ensuring the λ-optimality of a graph, one of which is that every minimally edge connected graph is λ-optimal ([7], exercise 49). A graph G is called *minimally edge connected* if $\lambda(G - e) < \lambda(G)$ for each edge $e \in E(G)$.

In 1988, Esfahanian and Hakimi proposed the concept of restricted edge connectivity [5,6]. An edge set $S \subset E$ is said to be a *restricted edge cut* if $G - S$ is disconnected and there is no isolated vertex in $G - S$. The *restricted edge connectivity* of G, denoted by $\lambda'(G)$, is the cardinality of a minimum restricted edge cut of G. It is proved in [6] that for any connected graph G of order at least 4 which is not isomorphic to the star $K_{1,n-1}$, $\lambda'(G)$ exists and satisfies $\lambda'(G) \leq \xi(G)$, where $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E\}$ is the *minimum edge degree* of G. It is shown by Wang and Li that the larger $\lambda'(G)$ is, the more reliable the networks is [10]. So, a graph G with $\lambda'(G) = \xi(G)$ is called a λ'-*optimal graph*. There is much research on λ'-optimal graphs (see for example [1,2,8,9,12,13]).

A graph G is called *minimally restricted edge connected* if $\lambda'(G-e) < \lambda'(G)$ for each edge $e \in E(G)$. It is implied in the definition that $\lambda'(G - e)$ exists for each edge e. So, we do not consider the case where there is a pending edge in G, and thus $\delta(G) \geq 2$. In this work, we show that every minimally restricted edge connected graph is λ'-optimal.

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Next, we introduce some terminologies used in this work.

A graph is *trivial* if it contains only one vertex; otherwise, it is *non-trivial*. For two disjoint vertex sets $U_1, U_2 \subset V(G)$, denote by $[U_1, U_2]$ the set of edges with one end in U_1 and the other end in U_2. $G[U]$ is the subgraph of G induced by the vertex set $U \subseteq V(G)$, $\overline{U} = V(G) \setminus U$ is the complement of U. Write $\omega(U) = |[U, \overline{U}]|$, and $d_U(u) = |[\{u\}, U \setminus \{u\}]|$. For simplicity of notation, we sometimes use a graph itself to represent its vertex set. For instance, $|[C, \overline{C}]|$ and $\omega(C)$ is used instead of $|[V(C), \overline{V(C)}]|$ and $\omega(V(C))$, where C is a subgraph of G. A vertex set U is called a λ'-*fragment* if $[U, \overline{U}]$ is a restricted edge cut with $\omega(U) = \lambda'(G)$. Obviously, if U is a λ'-fragment, so is $\overline{U}$. For a λ'-fragment U, $G[U]$ and $G[\overline{U}]$ are both connected by the minimality of $|[U, \overline{U}]|$. A λ'-fragment with the minimum cardinality is called a λ'-*atom*. The cardinality of a λ'-atom is denoted by $\alpha'(G)$. Clearly, $2 \leq \alpha'(G) \leq \frac{1}{2}|V(G)|$.

The following two observations will be frequently used without mentioning them explicitly. The first one is that if two connected subgraphs G_1 and G_2 have non-empty intersection, then $G_1 \cup G_2$ is also connected. The second one is that for a vertex set $F \subseteq V(G)$ and a component C of $G - F$, if $G[F]$ is connected, then $G - C$ is also connected.

The following submodular inequality plays an important role in studying various kinds of connectivities [11]: for two vertex sets $A, B \subset V$,

$$\omega(A \cap B) + \omega(A \cup B) \leq \omega(A) + \omega(B).$$

We follow [4] for terminologies and notation not given here.

2. Main result

Lemma 1. *Let $G = (V, E)$ be a λ'-connected graph, A be a λ'-atom of G, and B be a λ'-fragment of G. Suppose $\alpha'(G) \geq 3$. Then:*

(a) *$d_B(u) \geq d_{V \setminus B}(u)$ for each $u \in B$, except when $G[B]$ is a star and u is the center.*

(b) *$d_A(u) > d_{V \setminus A}(u)$ for each $u \in A$.*

(c) *$d_A(u) \geq 2$ holds for any vertex $u \in A$ with $d_G(u) \geq 2$. In particular, if $\delta(G) \geq 2$, then $\delta(G[A]) \geq 2$.*

Proof. Suppose there is a vertex $u \in B$ with $d_B(u) < d_{V \setminus B}(u)$. Furthermore, if $G[B]$ is a star, suppose u is not the center. Then, there is a non-trivial component C of $G[B] - u$. Note that $G[\overline{C}]$ is also connected since $G[\overline{B}]$ is connected, u is connected to $G[\overline{B}]$ for $d_{V \setminus B}(u) > 0$, and every other component of $G[B - u] - C$ is connected to u. So $[C, \overline{C}]$ is a restricted edge cut of G, and thus

$$\omega(C) \geq \lambda'(G). \tag{1}$$

On the other hand,

$$\omega(C) = |[C, \overline{B}]| + |[C, u]| \leq \omega(B) - d_{V \setminus B}(u) + d_B(u) < \omega(B) = \lambda'(G), \tag{2}$$

a contradiction.

The proof of (b) is similar to that of (a). The difference is that under the assumption $d_A(u) \leq d_{V \setminus A}(u)$, inequality (2) becomes $\omega(C) \leq \lambda'(G)$. Combining with inequality (1), we have $\omega(C) = \lambda'(G)$, and thus $V(C)$ is a smaller λ'-fragment than A, contradicting that A is a λ'-atom.

(c) is a consequence of (b). □

Lemma 2. *Let $G = (V, E)$ be a λ'-connected graph, A be a λ'-atom of G and B be a λ'-fragment of G. Suppose $A \cap B \neq \emptyset$, $A \setminus B \neq \emptyset$, $\overline{A \cup B} \neq \emptyset$, and $\alpha'(G) \geqslant 3$. Then,*

(a) *At least one of $A \cap B$ and $\overline{A \cup B}$ is an independent set.*

(b) *If there is a proper subset F of A with $|F| \geq 2$, $G[\overline{F}]$ being connected and $\omega(F) \leq \lambda'(G)$, then F is an independent set.*

Proof. (a) Suppose neither $A \cap B$ nor $\overline{A \cup B}$ is independent. Then there exist non-trivial components C of $G[A \cap B]$ and D of $G[\overline{A \cup B}]$. Clearly, $\omega(C) \leq \omega(A \cap B)$ and $\omega(D) \leq \omega(\overline{A \cup B}) = \omega(A \cup B)$. By noting that $G[C]$, $G[D]$, $G[\overline{C}]$ and $G[\overline{D}]$ are all connected, we have $\omega(C) \geq \lambda'(G)$ and $\omega(D) \geq \lambda'(G)$. So,

$$2\lambda'(G) \leq \omega(C) + \omega(D) \leq \omega(A \cup B) + \omega(A \cap B) \leq \omega(A) + \omega(B) = 2\lambda'(G).$$

It follows that $\omega(C) = \lambda'(G)$, and thus C is a λ'-fragment with fewer vertices than A, a contradiction.