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A numerical computation on the structure of the roots of q-extension of Genocchi polynomials

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Abstract

In this work we observe the behavior of real roots of the q-extension of Genocchi polynomials, $c_{n,q}(x)$, using numerical investigation. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the $c_{n,q}(x)$ for $-1 < q < 0$. Finally, we give a table for the solutions of the q-extension of Genocchi polynomials.

Keywords: Genocchi numbers; Genocchi polynomials; q-extension of Genocchi numbers; q-extension of Genocchi polynomials; Roots of the q-extension of Genocchi polynomials; Reflection symmetries of the q-extension of Genocchi polynomials

1. Introduction

In the 21st century, the computing environment will make more and more rapid progress. Over the years, there has been increasing interest in solving mathematical problems with the aid of computers. Recently, many mathematicians have studied Genocchi polynomials and Genocchi numbers. Genocchi polynomials and Genocchi numbers possess many interesting properties and arise in many areas of mathematics and physics. In [2], Kim constructed the q-extension of the Genocchi numbers $c_{n,q}$ and polynomials $c_{n,q}(x)$ using generating functions. In order to study the q-extension of Genocchi polynomials $c_{n,q}(x)$, we must understand the structure of the q-extension of Genocchi polynomials $c_{n,q}(x)$. Therefore, using computers, a realistic study for the q-extension of Genocchi polynomials $c_{n,q}(x)$ is very interesting. For related topics the interested reader is referred to [3]. The main purpose of this work is to describe the distribution and structure of the zeros of the q-extension of Genocchi polynomials for values of the index n by using computers. The outline of this work is as follows. We introduce the q-extension of Genocchi polynomials $c_{n,q}(x)$. In Section 2, we describe the beautiful zeros of the q-extension of Genocchi polynomials $c_{n,q}(x)$ using a numerical investigation. We also investigate the roots of the q-extension of Genocchi polynomials $c_{n,q}(x)$. Finally, we consider the reflection symmetries of the q-extension of Genocchi polynomials $c_{n,q}(x)$.

First, we introduce the Genocchi numbers and Genocchi polynomials. The Genocchi numbers G_n are defined by the generating function

$$F(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (|t| < \pi), \text{ cf. [1,2]} \tag{1}$$

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where we use the technique method notation by replacing G^n by G_n $(n \geq 0)$ symbolically. Here is the list of the first Genocchi numbers:

$$G_1 = 1, \quad G_2 = -1, \quad G_3 = 0, \quad G_4 = -1, \quad G_5 = 0, \quad G_6 = -3,$$
$$G_7 = 0, \quad G_8 = 17, \quad G_9 = 0, \quad G_{10} = -155, \quad G_{11} = 0, \quad G_{12} = 2073,$$
$$G_{14} = -38\,227 \quad G_{16} = 929\,569, \quad G_{18} = -28\,820\,619 \quad G_{20} = 1109\,652\,905, \ldots.$$

In general, it satisfies $G_3 = G_5 = G_7 = \cdots = 0$, and even coefficients are given by $G_n = 2(1-2^{2n})B_{2n} = 2nE_{2n-1}$, where B_n are Bernoulli numbers and E_n are Euler numbers.

For $x \in \mathbb{R}$ (=the field of real numbers), we consider the Genocchi polynomials $G_n(x)$ as follows:

$$F(x,t) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \tag{2}$$

Note that $G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}$. In the special case $x = 0$, we define $G_n(0) = G_n$. Here is the list of the first Genocchi polynomials:

$$G_1(x) = 1, \quad G_2(x) = 2x - 1, \quad G_3(x) = 3x^2 - 3x, \quad G_4(x) = 4x^3 + 6x^2 - 1$$
$$G_5(x) = 5x^4 - 10x^3 + 5x, \quad G_6(x) = 6x^5 - 15x^4 + 15x^2 - 3, \ldots.$$

Next, we introduce the q-extension of Genocchi polynomials $c_{n,q}(x)$ (see [1,2]). Some of the following consideration is the same as that of Kim [2] except for obvious modifications. Let q be a complex number with $|q| < 1$. We consider the following generating functions:

$$F_q(t) = \sum_{n=0}^{\infty} c_{n,q} \frac{t^n}{n!} = e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{(2n+1)(1-q^n)}{1-q^{2n+1}} \left(\frac{1}{1-q}\right)^{n-1} (-1)^{n-1} \frac{t^n}{n!}, \tag{3}$$

and

$$F_q(x,t) = \sum_{n=0}^{\infty} c_{n,q}(x) \frac{t^n}{n!} = e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{(2n+1)(1-q^n)}{1-q^{2n+1}} \left(\frac{1}{1-q}\right)^{n-1} q^{nx} (-1)^{n-1} \frac{t^n}{n!}. \tag{4}$$

By simple calculation in (4), we obtain

$$\begin{aligned}
F_q(x,t) &= e^{\frac{t}{1-q}} \sum_{i=0}^{\infty} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{i-1} (-1)^{i-1} q^{ix} \frac{t^i}{i!} \\
&= \left(\sum_{i=0}^{\infty} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{i-1} (-1)^{i-1} q^{ix} \frac{t^i}{i!} \right) \left(\sum_{j=0}^{\infty} \left(\frac{1}{1-q}\right)^j \frac{t^j}{j!} \right) \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{i-1} (-1)^{i-1} q^{ix} \frac{t^i}{i!} \left(\frac{1}{1-q}\right)^{n-i} \frac{t^{n-i}}{(n-i)!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{n-1} (-1)^{i-1} q^{ix} \right) \frac{t^n}{n!}.
\end{aligned}$$

For $n \geq 0$, we have

$$c_{n,q}(x) = \sum_{i=0}^{n} \binom{n}{i} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{n-1} (-1)^{i-1} q^{ix}.$$

When $x = 0$, we write $c_{n,q} = c_{n,q}(0)$, which are called the q-extension of Genocchi numbers. $c_{n,q}(x)$ is a polynomial of degree n in q^x. By definition of the q-extension of Genocchi polynomials $c_{n,q}(x)$, we obtain

$$c_{n,q} = \sum_{i=0}^{n} \binom{n}{i} \frac{(2i+1)(1-q^i)}{1-q^{2i+1}} \left(\frac{1}{1-q}\right)^{n-1} (-1)^{i-1}.$$