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# Some applications of a subordination theorem for a class of analytic functions

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**Abstract**

By making use of a subordination theorem for analytic functions, we derive several subordination relationships between certain subclasses of analytic functions which are defined by means of the Sălăgean derivative operator. Some interesting corollaries and consequences of our results are also considered.


*Keywords:* Analytic functions; Univalent functions; Subordinating factor sequence; Sălăgean derivative operator; Hadamard product (or convolution)

## 1. Introduction, definitions and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ $(0 \leqq \alpha < 1)$ the class of *starlike functions of order* $\alpha$ and the class of *convex functions of order* $\alpha$, respectively, where (see, for details, [2] and [4]; see also the references cited in *each* of these recent works)

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\quad 0 \leqq \alpha < 1) \right\}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U};\quad 0 \leqq \alpha < 1) \right\}.$$

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Clearly, we have

$$f(z) \in \mathcal{K}(\alpha) \iff zf'(z) \in \mathcal{S}^*(\alpha).$$

Sălăgean [5] introduced the following operator which is popularly known as the *Sălăgean derivative operator*:

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z)$$

and, in general,

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \ \mathbb{N} := \{1, 2, 3, \ldots\}).$$

We easily find from (1.1) that

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \quad (f \in \mathcal{A};\ n \in \mathbb{N}_0).$$

Let $\mathcal{N}_{m,n}(\alpha, \beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following inequality:

$$\Re\left(\frac{D^m f(z)}{D^n f(z)}\right) > \beta \left|\frac{D^m f(z)}{D^n f(z)} - 1\right| + \alpha \quad (z \in \mathbb{U};\ \ 0 \leqq \alpha < 1;\ \ \beta \geqq 0;\ \ m \in \mathbb{N};\ \ n \in \mathbb{N}_0).$$

Also let $\mathcal{M}^s_{m,n}(\alpha, \beta)$ $(s \in \mathbb{N}_0)$ be the subclasses of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following condition:

$$f(z) \in \mathcal{M}^s_{m,n}(\alpha, \beta) \Longleftrightarrow D^s f(z) \in \mathcal{N}_{m,n}(\alpha, \beta).$$

For $s = 0$, it is easily verified that

$$\mathcal{M}^0_{m,n}(\alpha, \beta) \equiv \mathcal{N}_{m,n}(\alpha, \beta).$$

The function classes $\mathcal{N}_{m,n}(\alpha, \beta)$ and $\mathcal{M}^s_{m,n}(\alpha, \beta)$ were introduced by Eker and Owa [1], who gave the following coefficient inequalities associated with these function classes.

**Theorem A** (*Eker and Owa [1]*). *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{j=2}^{\infty} \left\{\left|j^m - j^n - \alpha j^n\right| + \left(j^m + j^n - \alpha j^n\right) + 2\beta \left|j^m - j^n\right|\right\} \left|a_j\right| \leqq 2(1 - \alpha)$$
$$(0 \leqq \alpha < 1;\ \ \beta \geqq 0;\ \ m \in \mathbb{N};\ \ n \in \mathbb{N}_0), \tag{1.2}$$

*then $f(z) \in \mathcal{N}_{m,n}(\alpha, \beta)$.*

**Theorem B** (*Eker and Owa [1]*). *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$\sum_{j=2}^{\infty} j^s \left\{\left|j^m - j^n - \alpha j^n\right| + \left(j^m + j^n - \alpha j^n\right) + 2\beta \left|j^m - j^n\right|\right\} \left|a_j\right| \leqq 2(1 - \alpha)$$
$$(0 \leqq \alpha < 1;\ \ \beta \geqq 0;\ \ m \in \mathbb{N};\ \ n \in \mathbb{N}_0), \tag{1.3}$$

*then $f(z) \in \mathcal{M}^s_{m,n}(\alpha, \beta)$.*

In view of Theorems A and B, we now introduce the subclasses

$$\widehat{\mathcal{N}}_{m,n}(\alpha, \beta) \subset \mathcal{N}_{m,n}(\alpha, \beta) \quad \text{and} \quad \widehat{\mathcal{M}}^s_{m,n}(\alpha, \beta) \subset \mathcal{M}^s_{m,n}(\alpha, \beta),$$

which consist of functions $f(z) \in \mathcal{A}$ whose Taylor–Maclaurin coefficients satisfy the inequalities (1.2) and (1.3), respectively.

In this work, we prove several subordination relationships involving the function classes $\widehat{\mathcal{N}}_{m,n}(\alpha, \beta)$ and $\widehat{\mathcal{M}}^s_{m,n}(\alpha, \beta)$. In our proposed investigation of functions in these subclasses of the normalized analytic function class $\mathcal{A}$, we need the following definitions and results.