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Global entropy solutions to a variant of the compressible Euler equations

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Abstract

DiPerna [R.J. DiPerna, Global solutions to a class of nonlinear hyperbolic systems of equations, Comm. Rat. Pure Appl. Math. 26 (1973) 1–28] use the Glimm's scheme method to obtain a global weak solution to the Euler equations of one-dimensional, compressible fluid flow with $1 < \gamma < 3$, while in this work, we use the compensated compactness method coupled with some basic ideas of the kinetic formulation developed by Lions, Perthame, Souganidis and Tadmor [P.L. Lions, B. Perthame, P.E. Souganidis, Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, Comm. Pure Appl. Math. 49 (1996) 599–638; P.L. Lions, B. Perthame, E. Tadmor, Kinetic formulation of the isentropic gas dynamics and p-system, Comm. Math. Phys. 163 (1994) 415–431] to obtain the existence of global entropy solutions to the system with a uniform amplitude bound.

Keywords: Strong entropy; Entropy solution; Kinetic formulation

1. Introduction

Let us consider the Cauchy problem for the nonlinear hyperbolic system

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + \left(\frac{1}{2}u^2 + P(\rho)\right)_x = 0 \end{cases} \tag{1}$$

with bounded measurable initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x))\ \rho_0(x) \ge 0, \tag{2}$$

where the nonlinear function $P(\rho) = \frac{\theta}{2}\rho^{\gamma-1}$, $\theta = \frac{\gamma-1}{2}$ and $\gamma \in (1,3)$ is a constant.

System (1) was first derived by Earnshaw [2] in 1858 for isentropic flow and is also referred to as the Euler equations of one-dimensional, compressible fluid flow, where ρ denotes the density, u the velocity, and $P(\rho)$ the pressure of the fluid. System (1) has other different physical backgrounds. For instance, it is a scaling limit system of

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a Newtonian dynamics with long-range interaction for a continuous distribution of mass in R and also a hydrodynamic limit for the Vlasov equation (see [5]).

By simple calculations, two eigenvalues of system (1) are

$$\lambda_1 = u - \theta\rho^\theta, \qquad \lambda_2 = u + \theta\rho^\theta$$

with corresponding right eigenvectors

$$r_1 = (1, -\theta\rho^{\theta-1})^{\mathrm{T}}, \qquad r_1 = (1, \theta\rho^{\theta-1})^{\mathrm{T}};$$

the two corresponding Riemann invariants are

$$w = u + \rho^\theta, \qquad z = u - \rho^\theta;$$

and

$$\nabla\lambda_1 \cdot r_1 = -\theta(\theta+1)\rho^{\theta-1}, \qquad \nabla\lambda_2 \cdot r_2 = \theta(\theta+1)\rho^{\theta-1}.$$

Thus both characteristic fields are linearly degenerate on $\rho = \infty$, since $1 < \gamma < 3$.

The study of the existence of global weak solutions for the Cauchy problem (1) and (2) was started by DiPerna [1] for the case of $1 < \gamma < 3$ by using the Glimm's scheme method, while in this work, we use the compensated compactness method and the kinetic formulation to get the existence of global entropy solutions for the Cauchy problem with a uniform amplitude bound. Namely, we assume the viscosity solutions to the following Cauchy problem (3) and (4) for the related parabolic system are uniformly bounded,

$$\begin{cases} \rho_t + (\rho u)_x = \varepsilon\rho_{xx} \\ u_t + \left(\dfrac{1}{2}u^2 + P(\rho)\right)_x = \varepsilon u_{xx} \end{cases} \tag{3}$$

with initial data

$$(\rho(x,0), u(x,0)) = (\rho_0^\varepsilon(x), u_0^\varepsilon(x)), \tag{4}$$

where $(\rho_0^\varepsilon(x), u_0^\varepsilon(x)) = (\rho_0(x) + \varepsilon, u_0(x)) * G^\varepsilon$, and G^ε is a mollifier.

Theorem 1. *Let the initial data $(\rho_0(x), u_0(x))$ be bounded measurable and $\rho_0(x) \ge 0$. Then the Cauchy problem (1) and (2) with a uniform amplitude bound has a global bounded entropy solution.*

Remark 1. A pair of functions $(\rho(x,t), u(x,t))$ is called an entropy solution of the Cauchy problem (1) and (2) if

$$\begin{cases} \displaystyle\int_0^\infty \int_{-\infty}^\infty \rho\phi_t + \rho u\phi_x \mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty \rho_0(x)\phi(x,0)\mathrm{d}x = 0 \\ \displaystyle\int_0^\infty \int_{-\infty}^\infty u\phi_t + \left(\frac{1}{2}u^2 + P(\rho)\right)\phi_x \mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty u_0(x)\phi(x,0)\mathrm{d}x = 0 \end{cases}$$

for any test function $\phi(x,t) \in C_0^1(R \times R^+)$ and

$$\eta(\rho(x,t), u(x,t))_t + q(\rho(x,t), u(x,t))_x \le 0$$

in the sense of distributions for any convex entropy $\eta(\rho, u)$ of system (1), where $q(\rho, u)$ is the entropy flux associated with $\eta(\rho, u)$.

2. Proof of Theorem 1

Since the viscosity solutions to the Cauchy problem (3) and (4) are uniformly bounded, there exists a subsequence of the viscosity solutions (still labelled) $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$ such that

$$w^\star - \lim(\rho^\varepsilon(x,t), u^\varepsilon(x,t)) = (\rho(x,t), u(x,t)).$$

We shall show that $(\rho(x,t), u(x,t))$ is an entropy solution of the Cauchy problem (1) and (2). For simplicity, we will drop the superscript ε.